

# HAANTJES MANIFOLDS OF CLASSICAL INTEGRABLE SYSTEMS

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**ABSTRACT.** A general theory of classical integrable systems is proposed, based on the geometry of the Haantjes tensor. We introduce the class of symplectic-Haantjes manifolds (or  $\omega\mathcal{H}$  manifold), as the natural setting where the notion of integrability can be formulated. We prove that the existence of suitable Haantjes structures is a necessary and sufficient condition for a Hamiltonian system to be integrable in the Liouville-Arnold sense.

We also prove theorems ensuring the existence of a large family of completely integrable systems, constructed starting from a prescribed Haantjes structure.

Furthermore, we propose a novel approach to the theory of separation of variables, intimately related to the geometry of Haantjes manifolds. A special family of coordinates, that we shall call the Darboux-Haantjes coordinates, will be introduced. They are constructed from the Haantjes structure associated with an integrable system, and allow the additive separation of variables of the Hamilton-Jacobi equation.

Our analysis is performed in  $n$  degrees of freedom. We prove that some of the most classical examples of multidimensional Hamiltonian systems, as for instance the Gantmacher class, possess a natural Haantjes structure. Finally, we present an application of our approach to the study of some models, as a stationary reduction of the KdV hierarchy and a Drach-Holt type system; the separability properties of the latter were not known.

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## 1. INTRODUCTION

Integrable systems play a relevant role in modern science: they are ubiquitous in many branches of modern mathematics and theoretical physics. This is the reason motivating in the last decades the search for intrinsic mathematical structures underlying the notion of integrability. In particular, the investigation of the properties of exactly solvability of integrable systems led to the discovery of important analytic and geometric techniques. Finite-dimensional integrable models coming from classical or quantum mechanics, and the infinite-dimensional ones described in terms of soliton equations, share indeed many geometric and algebraic properties.

The study of the geometry of classical integrable systems has a long history, dating back to the works of Liouville, Jacobi, Stäckel, Eisenhart, Arnold, etc. In this context, the *bi-Hamiltonian* approach has proved to be crucial.

Essentially, a bi-Hamiltonian manifold is a differentiable manifold endowed with a pencil of Poisson structures [34]. In particular, the special class of  $\omega N$  manifolds, introduced in [43, 29], is characterized by a non-degenerate Poisson bivector (whose inverse provides a symplectic structure  $\omega$ ), and a compatible  $(1, 1)$  tensor field  $N$  with vanishing Nijenhuis torsion. Such a tensor, also called hereditary operator, has a vanishing Nijenhuis torsion, as a consequence of the underlying bi-Hamiltonian structure. The class of  $\omega N$  manifolds offers a coherent approach to the construction of separation variables, and has been successfully applied, for instance, to the study of Gelfand-Zakharevich systems [24, 30, 18, 19, 20].

The purpose of this paper is to present a new formulation of the notion of classical integrability, based on the theory of the *Haantjes tensor*. This tensor has been introduced in 1955 by Haantjes in [27], as a natural generalization of the Nijenhuis tensor. Quite surprisingly, the relevance of the Haantjes differential-geometric work in the realm of integrable systems has not been recognized for a long time, with the exception of some applications to Hamiltonian systems of hydrodynamic type [21, 11, 22].

We shall define a new family of manifolds, called *symplectic-Haantjes manifolds*. We shall prove that integrability of a finite-dimensional system can be characterized in terms of a set of commuting Haantjes operators, whose spectral and geometric properties turn out to be particularly rich. The notion of *Lenard-Haantjes chain*, defined in this framework, is a natural extension in the context of Haantjes geometry of previous similar notions known in the literature, as that of Lenard-Magri chain

[35] and of generalized Lenard chain [60, 17, 37], for quasi-bi-Hamiltonian systems and their generalizations.

There is a neat relation between the Haantjes geometry developed here and the known Nijenhuis geometry. In fact, a subfamily of symplectic-Haantjes manifolds is provided by the class of symplectic-Nijenhuis manifolds. Precisely, we shall show that given a  $\omega\mathbf{N}$  manifold, one can construct, under mild assumptions, a  $\omega\mathcal{H}$  structure by taking  $(n - 1)$  independent powers of  $\mathbf{N}$ . In this case,  $\mathbf{N}$  will play the role of a generator of a  $\omega\mathcal{H}$  structure.

The notion of  $\omega\mathcal{H}$  manifolds is inspired by the very recent definition of Haantjes manifold due to Magri [38, 39, 40, 41]. Our theory mainly differs from the fact that, besides the existence of  $n$  independent commuting Haantjes operators, we also allow a symplectic form  $\omega$  compatible with the Haantjes operators to exist. Moreover, the Lenard-Haantjes chains of our theory are shorter than the ones defined in the recent Magri's theory [39]. This is due to a weaker assumption that allows us to deal with both integrable and separable systems. This fact is an important novelty of the present work that is not considered in the cited papers.

Our main result concerning integrability is a theorem establishing that *the existence of a  $\omega\mathcal{H}$  manifold is a necessary and sufficient condition for a non-degenerate Hamiltonian system to be integrable* in the Liouville-Arnold sense. As a byproduct of this theorem, we will be able to define new general classes of integrable models possessing an assigned Haantjes geometry. Quite interestingly, the systems so obtained are related to analytic functions and to the wave equation.

Another advantage of the present formulation *à la Haantjes* (which also represents the main motivation for our study), is its generality: Haantjes tensors are indeed a larger class of tensors than those of Nijenhuis. The proposed theory incorporates essentially all the known results on integrability and separation of variables of finite-dimensional systems that have been developed in a bi-Hamiltonian framework up to date, i.e. all the approaches based on Lenard chains and their generalizations (as quasi-bi-Hamiltonian systems [8], etc).

A noteworthy aspect is that the Haantjes operators appearing in the theory are *not necessarily diagonalizable*. This aspect represents a significant generalization of the  $\omega\mathbf{N}$  approach, where the operator  $\mathbf{N}$  is diagonalizable by hypothesis. Moreover, our theory keeps the intrinsic simplicity enjoyed by the standard approach to the Lenard-Magri chains for soliton hierarchies.

At the same time, the theory of  $\omega\mathcal{H}$  manifolds is motivated by the crucial problem of the construction of coordinate systems allowing the additive separation of the associated Hamilton-Jacobi (HJ) equation (the separation variables). This is, perhaps, the most important problem in the theory of classical integrable systems, to which many important contributions have been made.

In 1904, Levi-Civita proposed in [33] a test for verifying whether a given Hamiltonian is separable in an assigned coordinate system. Another important result, due to Benenti [4], states that a family of Hamiltonian functions  $\{H_i\}_{1 \leq i \leq n}$  is separable in a set of canonical coordinates  $(\mathbf{q}, \mathbf{p})$  if and only if they are in separable involution, i.e. if and only if they satisfy the relations

$$(1) \quad \{H_i, H_j\}_{|k} = \frac{\partial H_i}{\partial q_k} \frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k} \frac{\partial H_j}{\partial q_k} = 0, \quad 1 \leq k \leq n,$$

where no summation over  $k$  is understood. However, such a theorem as well as the Levi-Civita test are not constructive, since they do not help to find a complete

integral of the Hamilton–Jacobi equation. By contrast, a constructive definition of separation of variables (SoV) was given by Sklyanin [57] within the framework of Lax systems. In this setting, the Hamiltonian functions  $\{H_i\}_{1 \leq i \leq n}$  are separable in a set of canonical coordinates  $(\mathbf{q}, \mathbf{p})$  if there exist  $n$  equations, called separation relations, of the form

$$(2) \quad \Phi_i(q_i, p_i; H_1, \dots, H_n) = 0 \quad \det \left[ \frac{\partial \Phi_i}{\partial H_j} \right] \neq 0,$$

for  $i = 1, \dots, n$ . They are named the Jacobi-Sklyanin separation equations for  $\{H_i\}_{1 \leq i \leq n}$ . The Jacobi-Sklyanin equations allow to construct a solution  $(W, E)$  of the HJ equation. In fact, by solving (2) with respect to  $p_k = \frac{\partial W_k}{\partial q_k}$ , one gets

$$(3) \quad W = \sum \int p_k(q'_k; H_1, \dots, H_n)|_{H_i=a_i} dq'_k.$$

However, the three above-mentioned criteria of separability are not intrinsic, since they require the explicit knowledge of the local chart  $(\mathbf{q}, \mathbf{p})$  in order to be applied. To overcome such a drawback, in the last decades several approaches based on symplectic and Poisson geometry have been designed; they have offered a fundamental geometric insight into the theory of integrable systems. At the same time, a geometric version of integrability on differentiable manifolds can share new light on the multiple connections among integrability, topological field theories, singularity theory, co-isotropic deformations of associative algebras, etc.

The problem of SoV can be recast and completely solved, in principle, in our approach. With respect to this problem, the main achievement is a theorem ensuring the existence, under mild hypotheses, of a set of distinguished coordinates from the Haantjes structure associated with an integrable system, that we shall call the *Darboux-Haantjes coordinates*. They represent separation coordinates for the Hamilton-Jacobi equation associated with the system.

Besides, we shall prove that a huge class of very general, *multidimensional* separable systems indeed possess a Haantjes geometry. The so-called telescopic systems and the Gantmacher systems are all examples of very basic integrable models in  $n$  arbitrary dimensions possessing a  $\omega\mathcal{H}$  structure. Conversely, by using the underlying Haantjes structure, we will be able to determine separation coordinates for many integrable systems, including a family of Drach-Holt type systems, whose separability properties were not known.

We also mention that the fundamental class of generalized Stäckel systems and the relevant example of the Jacobi-Calogero model have been studied in [63] as an application of the theory proposed in the present work.

The structure of the paper is the following. In Section 2, we review the main algebraic properties of Nijenhuis and Haantjes tensors. In Section 3, the spectral properties of Haantjes operators are discussed. In Section 4, we introduce the main geometrical structures needed for the discussion of integrability, i.e. the  $\omega\mathcal{H}$  manifolds; also, we clarify their relation with  $\omega\mathcal{N}$  manifolds. Section 5 contains the theorem that characterizes complete integrability via the Haantjes geometry. In Section 6, new integrable models related to analytic functions and to the wave equation are deduced from suitable Haantjes structures. Section 7 is devoted to the problem of separability of the Hamilton-Jacobi equation in the context of Haantjes geometry. In particular, a theorem guaranteeing the existence of the DH coordinates is proved. In Section 8, a procedure for the construction of Haantjes structures

for a given integrable system with two degrees of freedom is proposed. Also, the relevant example of the superintegrable Post-Winternitz system, whose separation coordinates are still not known, is worked out. Some applications of our theory of separation of variables are proposed in Section 9. Open problems and future research plans are sketched in the final Section 10.

## 2. NIJENHUIS AND HAANTJES OPERATORS

The integrability of a dynamical system defined over a manifold  $M$  essentially amounts to find privileged coordinate webs in which the equations of motion decouple. The natural frames of such webs, being obviously integrable, can be characterized in a *tensorial manner* as eigen-distributions of a suitable class of  $(1, 1)$  tensor fields, i.e. the ones with vanishing Nijenhuis or Haantjes tensor. In this section, we review some basic algebraic results concerning the theory of such tensors. For a more complete treatment, see the original papers [27, 52] and the related ones [53, 23].

Let  $M$  be a differentiable manifold and  $\mathbf{L} : TM \rightarrow TM$  be a  $(1, 1)$  tensor field, i.e., a field of linear operators on the tangent space at each point of  $M$ .

**Definition 1.** *The Nijenhuis torsion of  $\mathbf{L}$  is the skew-symmetric  $(1, 2)$  tensor field defined by*

$$(4) \quad \mathcal{T}_{\mathbf{L}}(X, Y) := \mathbf{L}^2[X, Y] + [\mathbf{L}X, \mathbf{L}Y] - \mathbf{L}([X, \mathbf{L}Y] + [\mathbf{L}X, Y]),$$

where  $X, Y \in TM$  and  $[ , ]$  denotes the commutator of two vector fields.

In local coordinates  $\mathbf{x} = (x_1, \dots, x_n)$ , the Nijenhuis torsion can be written in the form

$$(5) \quad (\mathcal{T}_{\mathbf{L}})^i_{jk} = \sum_{\alpha=1}^n \left( \frac{\partial \mathbf{L}_k^i}{\partial x^\alpha} \mathbf{L}_j^\alpha - \frac{\partial \mathbf{L}_j^i}{\partial x^\alpha} \mathbf{L}_k^\alpha + \left( \frac{\partial \mathbf{L}_j^\alpha}{\partial x^k} - \frac{\partial \mathbf{L}_k^\alpha}{\partial x^j} \right) \mathbf{L}_\alpha^i \right),$$

amounting to  $n^2(n-1)/2$  independent components.

**Definition 2.** *The Haantjes tensor associated with  $\mathbf{L}$  is the  $(1, 2)$  tensor field defined by*

$$(6) \quad \mathcal{H}_{\mathbf{L}}(X, Y) := \mathbf{L}^2 \mathcal{T}_{\mathbf{L}}(X, Y) + \mathcal{T}_{\mathbf{L}}(\mathbf{L}X, \mathbf{L}Y) - \mathbf{L}(\mathcal{T}_{\mathbf{L}}(X, \mathbf{L}Y) + \mathcal{T}_{\mathbf{L}}(\mathbf{L}X, Y)).$$

The skew-symmetry of the Nijenhuis torsion implies that the Haantjes tensor is also skew-symmetric. Its local expression is

$$(7) \quad (\mathcal{H}_{\mathbf{L}})^i_{jk} = \sum_{\alpha, \beta=1}^n \left( \mathbf{L}_\alpha^i \mathbf{L}_\beta^\alpha (\mathcal{T}_{\mathbf{L}})_{jk}^\beta + (\mathcal{T}_{\mathbf{L}})_{\alpha\beta}^i \mathbf{L}_j^\alpha \mathbf{L}_k^\beta - \mathbf{L}_\alpha^i \left( (\mathcal{T}_{\mathbf{L}})_{\beta k}^\alpha \mathbf{L}_j^\beta + (\mathcal{T}_{\mathbf{L}})_{j\beta}^\alpha \mathbf{L}_k^\beta \right) \right).$$

We shall first consider some specific cases, in which the construction of the Nijenhuis and Haantjes tensors will be particularly simple.

**Example 3.** *Let  $\mathbf{L}$  be a field of operators that assumes a diagonal representation*

$$(8) \quad \mathbf{L} = \sum_{i=1}^n l_i(\mathbf{x}) \frac{\partial}{\partial x_i} \otimes dx_i,$$

in some local chart  $\mathbf{x} = (x_1, \dots, x_n)$ . Its Nijenhuis torsion is given by

$$(9) \quad (\mathcal{T}_{\mathbf{L}})_{jk}^i = (l_j - l_k) \left( \frac{\partial l_j}{\partial x_k} \delta_j^i + \frac{\partial l_k}{\partial x_j} \delta_k^i \right).$$

It is evident that  $(\mathcal{T}_{\mathbf{L}})_{jk}^i = 0$  if  $i \neq j$  and  $i \neq k$  or if  $j = k$ . Thus, we can limit ourselves to analyze the  $n(n-1)$  components

$$(10) \quad (\mathcal{T}_{\mathbf{L}})_{jk}^j = (l_j - l_k) \frac{\partial l_j}{\partial x_k}, \quad j \neq k.$$

If  $\frac{\partial l_j}{\partial x_k} \neq 0$ , each component vanishes if and only if  $l_j(\mathbf{x}) \equiv l_k(\mathbf{x})$ . Therefore, we can state the following

**Lemma 4.** *Let  $\mathbf{L}$  be the diagonal field of operators (8), and suppose that its Nijenhuis torsion vanishes. Let us denote with  $(i_1, \dots, i_j, \dots, i_r)$ ,  $r \leq n$  a subset of  $(1, 2, \dots, n)$ . If the  $j$ -th eigenvalue of  $\mathbf{L}$  depends on the variables  $(i_1, \dots, i_j, \dots, i_r)$ , then*

$$(11) \quad l_j(i_1, \dots, i_j, \dots, i_r) \equiv l_{i_1} \equiv l_{i_2} \equiv \dots \equiv l_{i_r}$$

Thus, apart when each eigenvalue is constant, we can distinguish several cases, ensuring that the Nijenhuis torsion of a diagonal operator vanishes. For instance,

- i)  $l_j(\mathbf{x}) = \lambda_j(x_j)$ ,  $j = 1, \dots, n \Rightarrow n$  simple eigenvalues
- ii)  $l_j(\mathbf{x}) = \lambda(\mathbf{x})$ ,  $j = 1, \dots, n \Rightarrow 1$  eigenvalue of multiplicity  $n$ ,

represent the extreme cases. An exhaustive analysis of all intermediate possibilities is left to the reader.

**Example 5.** *Let  $\dim M = 2$ . Then, it is easy to prove by a straightforward computation that the Haantjes tensor of any field of smooth operators vanishes.*

**Example 6.** *Let  $\mathbf{L}$  be the diagonal operator of Example 3. Its Haantjes tensor reads*

$$(12) \quad (\mathcal{H}_{\mathbf{L}})_{jk}^i = (l_i - l_j)(l_i - l_k)(\mathcal{T}_{\mathbf{L}})_{jk}^i,$$

where  $(\mathcal{T}_{\mathbf{L}})_{jk}^i$  is given by eq. (9).

The following proposition is a direct consequence of eqs. (9) and (12).

**Proposition 7.** *Let  $\mathbf{L}$  a smooth field of operators. If there exists a local coordinate chart  $\{(x_1, \dots, x_n)\}$ , where  $\mathbf{L}$  assumes the diagonal form (8), then the Haantjes tensor of  $\mathbf{L}$  vanishes.*

Due to the relevance of the Haantjes (Nijenhuis) vanishing condition, we propose the following definition.

**Definition 8.** *A Haantjes (Nijenhuis) field of operators is a field of operators whose Haantjes (Nijenhuis) tensor identically vanishes.*

As usual, the transposed operator  $\mathbf{L}^T : T^*M \mapsto T^*M$  is defined as the transposed linear map of  $\mathbf{L}$  with respect to the natural pairing between a vector space and its dual space

$$< \mathbf{L}^T \alpha, X > = < \alpha, \mathbf{L} X > \quad \alpha \in T^*M, X \in TM.$$

A relevant property of Nijenhuis operators, which is a direct consequence of Theor. 17 below and eq. (11), and usually is not satisfied by Haantjes operators, is the following

**Proposition 9.** *The differentials of the eigenvalues  $\lambda_i(\mathbf{x})$  of a diagonalizable Nijenhuis operator  $\mathbf{N}$  are eigenforms of its transposed operator  $\mathbf{N}^T$*

$$(13) \quad \mathbf{N}^T d\lambda_i = \lambda_i d\lambda_i .$$

Very similar statements can be found in [52] and [25].

The product of a Nijenhuis operator with a generic function is no longer a Nijenhuis operator, as is proved by the following identity

$$(14) \quad \mathcal{T}_f \mathbf{L}(X, Y) = f^2 \mathcal{T}_{\mathbf{L}}(X, Y) + f \left( (\mathbf{L}X)(f) \mathbf{L}Y - (\mathbf{L}Y)(f) \mathbf{L}X + Y(f) \mathbf{L}^2 X - X(f) \mathbf{L}^2 Y \right) ,$$

which holds for any function  $f \in C^\infty(M)$ . Instead, the differential and algebraic properties of a Haantjes operator are much richer, as follows from these remarkable results.

**Proposition 10.** [9]. *Let  $\mathbf{L}$  be a field of operators. The following identity holds*

$$(15) \quad \mathcal{H}_{f\mathbf{I}+g\mathbf{L}}(X, Y) = g^4 \mathcal{H}_{\mathbf{L}}(X, Y) ,$$

where  $f, g : M \rightarrow \mathbb{R}$  are  $C^\infty(M)$  functions, and  $\mathbf{I}$  denotes the identity operator in  $TM$ .

*Proof.* See Proposition 1, p. 255 of [9]. □

**Proposition 11.** [10]. *Let  $\mathbf{L}$  be an operator with vanishing Haantjes tensor in  $M$ . Then for any polynomial in  $\mathbf{L}$  with coefficients  $a_j \in C^\infty(M)$ , the associated Haantjes tensor vanishes, i.e.*

$$(16) \quad \mathcal{H}_{\mathbf{L}}(X, Y) = 0 \implies \mathcal{H}_{(\sum_j a_j(\mathbf{x}) \mathbf{L}^j)}(X, Y) = 0 .$$

*Proof.* See Corollary 3.3, p. 1136 of [10]. □

Propositions 10 and 11 imply that the powers of a single Haantjes operator (by contrast with the case of a Nijenhuis operator) generate a module over the ring of smooth functions on  $M$ .

Let us introduce an interesting example of Nijenhuis and Haantjes operators drawn from the realm of Rational Mechanics.

**Example 12.** *Let  $\mathcal{M} = \{(P_\gamma, m_\gamma) \in (\mathcal{E}_n, \mathbb{R})\}$  be a finite system of mass points (possibly with  $m_\gamma < 0$ ) in the  $n$ -dimensional affine Euclidean space  $\mathcal{E}_n$ . Let us consider the  $(1, 1)$  tensor field defined by*

$$(17) \quad \mathbf{E}_P(\vec{v}) = \sum_{\gamma} m_{\gamma} ((P_{\gamma} - P) \cdot \vec{v}) (P_{\gamma} - P) \quad \vec{v} \in T_P \mathcal{E}_n \equiv \mathbb{E}_n ,$$

called the planar inertia tensor (or Euler tensor in Continuum Mechanics), and the inertia tensor field, given by

$$(18) \quad \mathbb{I}_P(\vec{v}) = \sum_{\gamma} m_{\gamma} \left( |P_{\gamma} - P|^2 \vec{v} - ((P_{\gamma} - P) \cdot \vec{v}) (P_{\gamma} - P) \right) .$$

They are related by the formulas

$$(19) \quad \mathbb{I}_P = \text{trace}(\mathbf{E}_P) \mathbf{I}_n - \mathbf{E}_P , \quad \mathbf{E}_P = \frac{\text{trace}(\mathbb{I}_P)}{n-1} \mathbf{I}_n - \mathbb{I}_P ,$$

where  $\mathbf{I}_n$  is the identity operator in  $\mathbb{E}_n$ . Both of them are symmetric w.r.t. the Euclidean scalar product, so that they are diagonalizable at any point of  $\mathcal{E}_n$ . Furthermore, by virtue of (19) they commute; consequently, they can be simultaneously diagonalized.

If  $G$  is the center of mass of  $\mathcal{M}$ , defined by

$$G - P = \frac{1}{m} \sum_{\gamma} (P_{\gamma} - P) \quad m := \sum_{\gamma} m_{\gamma} \quad m \in \mathbb{R} \setminus \{0\},$$

the following Huygens-Steiner transposition formulas hold

$$(20) \quad \mathbf{E}_P(\vec{v}) = \mathbf{E}_G(\vec{v}) + m((P - G) \cdot \vec{v})(P - G),$$

$$(21) \quad \mathbb{I}_P(\vec{v}) = \mathbb{I}_G(\vec{v}) + m|P - G|^2 - m((P - G) \cdot \vec{v})(P - G).$$

From eqs. (20) and (21) it follows that in the Cartesian coordinates  $(x_1, \dots, x_n)$  with origin in  $G$ , defined by the common eigendirections of  $\mathbf{E}_G$  and  $\mathbb{I}_G$ , we have

$$(22) \quad (\mathbf{E}_P)^i_j = \lambda_i(G)\delta_{ij} + m x_i x_j,$$

$$(23) \quad (\mathbb{I}_P)^i_j = l_i(G)\delta_{ij} + m \left( \sum_{\alpha=1}^n x_{\alpha}^2 - x_i x_j \right),$$

where  $\lambda_i(G)$  and  $l_j(G)$  denote, respectively, the eigenvalues of the tensor fields  $\mathbf{E}$  and  $\mathbb{I}$ , both evaluated at the point  $G$ . In [5, 6] it has been proved that the Nijenhuis torsion of  $\mathbf{E}$  vanishes; then its Haantjes tensor also vanishes. Furthermore, we observe that the torsion of  $\mathbb{I}$  reads

$$(24) \quad (\mathcal{T}_{\mathbb{I}})^i_{jk} = 2m \sum_{\alpha=1}^n \left( x_{\alpha} \mathbb{I}^{\alpha}(\delta_{ik} - \delta_{ij}) + x_k \mathbb{I}^i_j - x_j \mathbb{I}^i_k \right),$$

i.e. it is not identically zero, although its Haantjes tensor vanishes as a consequence of the identity (15), applied to the relation (19).

Other relevant examples of Haantjes operators that are also Killing tensors in a Riemannian manifold can be found in [63].

### 3. THE GEOMETRY OF HAANTJES OPERATORS

As we noted in Proposition 7, the Haantjes tensor  $\mathcal{H}_{\mathbf{L}}$  of an operator  $\mathbf{L}$  has a relevant geometrical meaning: its vanishing is a necessary condition for the *eigen-distributions* of  $\mathbf{L}$  to be integrable. To clarify this point, let us recall that a reference frame is a set of  $n$  vector fields  $\{Y_1, \dots, Y_n\}$  such that, at each point  $\mathbf{x}$  belonging to an open set  $U \subseteq M$ , they form a basis of the tangent space  $T_{\mathbf{x}}U$ . Two frames  $\{X_1, \dots, X_n\}$  and  $\{Y_1, \dots, Y_n\}$  are said to be equivalent if  $n$  nowhere vanishing smooth functions  $f_i$  do exist such that

$$X_i = f_i(\mathbf{x})Y_i, \quad i = 1, \dots, n.$$

A natural frame is the frame associated to a local chart  $\{(x_1, \dots, x_n)\}$  and denoted as  $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ .

**Definition 13.** A holonomic frame is a reference frame equivalent to a natural frame.



In other words, to say that a frame  $\{Y_1, \dots, Y_n\}$  is holonomic there must exist  $n$  nowhere vanishing functions  $f_i$  and a local chart  $(x_1, \dots, x_n)$  such that

$$(25) \quad Y_i = f_i(\mathbf{x}) \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n.$$

**Proposition 14.** [7] *A reference frame in a manifold  $M$  is a holonomic frame if and only if it satisfies one the two equivalent conditions:*

- *each two-dimensional distribution generated by any two vector fields  $Y_i, Y_j$  is Frobenius integrable;*
- *each  $(n - 1)$ -dimensional distribution  $E_i$  generated by all the vector fields except  $Y_i$  is Frobenius integrable.*

**Definition 15.** *A field of operators  $\mathbf{L}$  is said to be semisimple (or diagonalizable) if there exists a reference frame formed by (proper) eigenvector fields of  $\mathbf{L}$ . This frame will be called an eigen-frame of  $\mathbf{L}$ . Moreover,  $\mathbf{L}$  is said to be simple if all its eigenvalues are pointwise distinct, namely if  $l_i(\mathbf{x}) \neq l_j(\mathbf{x})$ ,  $i, j = 1, \dots, n$ ,  $\forall \mathbf{x} \in M$ .*

Proposition 7 amounts to say that if an operator admits a local chart in which it takes a diagonal form, then its Haantjes tensor necessarily vanishes, therefore the associate coordinate frame is an eigenframe that is trivially holonomic. In 1955, Haantjes proved in [27] that the vanishing of the Haantjes tensor of a *semisimple* operator  $\mathbf{L}$  is also a sufficient condition to ensure the integrability of each of its eigen-distributions (with constant rank) and the existence of local coordinate charts in which  $\mathbf{L}$  takes a *diagonal* form. We call such coordinates *Haantjes coordinates*. Furthermore, he stated that the vanishing of the Haantjes tensor of an operator  $\mathbf{L}$  is also a sufficient (but not necessary) condition to ensure the integrability of each of its *generalized* eigen-distributions (with constant rank) and the existence of local coordinate charts in which  $\mathbf{L}$  takes a *Jordan* form. An equivalent statement of the above-mentioned results is that a Haantjes field of operators admits a generalized eigen-frame that is a *holonomic* frame.

Let us denote with  $\text{Spec}(\mathbf{L}) := \{l_1(\mathbf{x}), l_2(\mathbf{x}), \dots, l_s(\mathbf{x})\}$  the set of the distinct eigenvalues of an operator  $\mathbf{L}$ , which we always assume real in all the forthcoming considerations, and with

$$(26) \quad \mathcal{D}_i = \text{Ker}(\mathbf{L} - l_i(\mathbf{x})\mathbf{I})^{\rho_i}, \quad i = 1, \dots, s$$

the  $i$ -th generalized eigen-distribution, that is the distribution of all the generalized eigenvector fields corresponding to the eigenvalue  $l_i(\mathbf{x})$ . In eq. (26),  $\rho_i$  denotes the Riesz index of  $l_i$ , namely the minimum integer such that

$$(27) \quad \text{Ker}(\mathbf{L} - l_i(\mathbf{x})\mathbf{I})^{\rho_i} \equiv \text{Ker}(\mathbf{L} - l_i(\mathbf{x})\mathbf{I})^{\rho_i+1}.$$

When  $\rho_i = 1$ ,  $\mathcal{D}_i$  is a proper eigen-distribution.

**Definition 16.** *A generalized eigen-frame (or a Jordan eigen-frame) of a field of operators  $\mathbf{L}$  is a frame of generalized eigenvectors of  $\mathbf{L}$ .*

**Theorem 17.** [27]. *Let  $\mathbf{L}$  be a field of operators, and assume that the rank of each generalized eigen-distribution  $\mathcal{D}_i$  is independent of  $\mathbf{x} \in M$ . The vanishing of the Haantjes tensor*

$$(28) \quad \mathcal{H}_{\mathbf{L}}(X, Y) = 0 \quad \forall X, Y \in TM$$

is a sufficient condition to ensure the integrability of each generalized eigen-distribution  $\mathcal{D}_i$  and of any direct sum  $\mathcal{D}_i \oplus \mathcal{D}_j \oplus \dots \oplus \mathcal{D}_k$  (where all indices  $i, j, \dots, k$  are different). In addition, if  $\mathbf{L}$  is semisimple, the condition (28) is also necessary.

In the original paper by Haantjes, the proof of Theorem 17 is explicitly made only for the case of a semisimple operator. Below, we present the proof for the more general case of an operator admitting *generalized* eigenvectors with arbitrary Riesz index.

Here, for the sake of simplicity, we focus only on two eigenvalues of  $\mathbf{L}$ ,  $\mu$  and  $\nu$ , possibly coincident. Let us denote by  $X_\alpha, Y_\beta$ , two fields of generalized eigenvectors, with index  $\alpha$ , respectively  $\beta$ , corresponding to the eigenvalues  $\mu$ , resp.  $\nu$ , and belonging to a Jordan chain in  $\mathcal{D}_\mu$ , resp.  $\mathcal{D}_\nu$ ,

$$(29) \quad \mathbf{L}X_\alpha = \mu X_\alpha + X_{\alpha-1}, \quad \mathbf{L}Y_\beta = \nu Y_\beta + Y_{\beta-1}, \quad 1 \leq \alpha \leq \rho_\mu, \quad 1 \leq \beta \leq \rho_\nu,$$

where  $X_0$  and  $Y_0$  are the null vector fields. Then, it holds true that

$$(30) \quad X_\alpha \in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\rho_\mu}, \quad Y_\beta \in \text{Ker}(\mathbf{L} - \nu \mathbf{I})^{\rho_\nu}.$$

Evaluating the Nijenhuis torsion on such eigenvector fields, we get

$$\begin{aligned} \mathcal{T}_{\mathbf{L}}(X_\alpha, Y_\beta) &= (\mathbf{L} - \mu \mathbf{I})(\mathbf{L} - \nu \mathbf{I})[X_\alpha, Y_\beta] + (\mu - \nu)(X_\alpha(\nu)Y_\beta + Y_\beta(\mu)X_\alpha) \\ &- (\mathbf{L} - \mu \mathbf{I})[X_\alpha, Y_{\beta-1}] - (\mathbf{L} - \nu \mathbf{I})[X_{\alpha-1}, Y_\beta] + [X_{\alpha-1}, Y_{\beta-1}] \\ &- (X_\alpha(\nu)Y_{\beta-1} + Y_{\beta-1}(\mu)X_\alpha) + (X_{\alpha-1}(\nu)Y_\beta + Y_\beta(\mu)X_{\alpha-1}), \end{aligned}$$

where  $X_\alpha(\nu)$  denotes the Lie derivative of the eigenvalue  $\nu(x)$  with respect to the vector field  $X_\alpha$ . The analogous relation for the Haantjes tensor is

$$(31) \quad \mathcal{H}_{\mathbf{L}}(X_\alpha, Y_\beta) = \sum_{i,j=0}^2 (-1)^{i+j} \binom{2}{i} \binom{2}{j} (\mathbf{L} - \mu \mathbf{I})^{2-i} (\mathbf{L} - \nu \mathbf{I})^{2-j} [X_{\alpha-i}, Y_{\beta-j}].$$

**Lemma 18.** *Let  $\mathbf{L}$  be a field of operators and  $X_\alpha, Y_\beta$ , be two of its fields of generalized eigenvectors in  $\mathcal{D}_\mu$ , belonging to possibly different Jordan chains. If*

$$(32) \quad \mathcal{H}_{\mathbf{L}}(\mathcal{D}_\mu, \mathcal{D}_\mu) = 0,$$

*then their commutator satisfies the relation*

$$(33) \quad [X_\alpha, Y_\beta] \in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\alpha+\beta+2} \equiv \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\min(\alpha+\beta+2, \rho_\mu)} \subseteq \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\rho_\mu},$$

*where  $\min(\cdot, \cdot)$  means the minimum of its arguments.*

*Proof.* If  $\alpha = \beta = 1$  and  $\mu = \nu$ , eq. (31) implies that  $[X_1, Y_1] \in (\mathbf{L} - \mu \mathbf{I})^4$ . By induction over  $(\alpha + \beta)$ , and applying the operator  $(\mathbf{L} - \mu \mathbf{I})^{\alpha+\beta-2}$  to both members of eq. (31) it follows that  $[X_\alpha, Y_\beta] \in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\alpha+\beta+2}$ .  $\square$

**Proposition 19.** *Let  $\mathbf{L}$  be a field of operators. An eigen-distribution  $\mathcal{D}_\mu$  with Riesz index  $\rho_\mu$  is integrable if*

$$(34) \quad \mathcal{H}_{\mathbf{L}}(\mathcal{D}_\mu, \mathcal{D}_\mu) = 0.$$

*In particular, if  $\rho_\mu = 1$ , the converse is also true.*

*Proof.* Lemma 18 immediately implies that the Frobenius integrability condition for  $\mathcal{D}_\mu$

$$(35) \quad [\mathcal{D}_\mu, \mathcal{D}_\mu] \subseteq \mathcal{D}_\mu$$

is fulfilled. In particular, if  $\rho_\mu = 1$ , every  $\mu$ -eigenvector of  $\mathbf{L}$  is a proper eigenvector, and from eq. (31) one infers that

$$\mathcal{H}_{\mathbf{L}}(\mathcal{D}_\mu, \mathcal{D}_\mu) = 0 \iff [X_1, Y_1] \in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^4 \equiv \text{Ker}(\mathbf{L} - \mu \mathbf{I}) = \mathcal{D}_\mu.$$

□

**Lemma 20.** *Let  $\mathbf{L}$  be a Haantjes operator. The commutator of two generalized eigenvector fields of  $\mathbf{L}$ , with different eigenvalues  $\mu, \nu$ , fulfills the relation*

$$(36) \quad \begin{aligned} [X_\alpha, Y_\beta] &\in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\alpha+1} \oplus \text{Ker}(\mathbf{L} - \nu \mathbf{I})^{\beta+1} \\ &\equiv \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\min(\alpha+1, \rho_\mu)} \oplus \text{Ker}(\mathbf{L} - \nu \mathbf{I})^{\min(\beta+1, \rho_\nu)} \\ &\subseteq \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\rho_\mu} \oplus \text{Ker}(\mathbf{L} - \nu \mathbf{I})^{\rho_\nu}, \end{aligned}$$

with  $1 \leq \alpha \leq \rho_\mu, 1 \leq \beta \leq \rho_\nu$ .

*Proof.* If  $\alpha = \beta = 1$  and  $\mu \neq \nu$ , eq. (31) implies that  $[X_1, Y_1] \in \text{Ker}(\mathbf{L} - \mu \mathbf{I})^2 \oplus \text{Ker}(\mathbf{L} - \nu \mathbf{I})^2$ . By induction over  $(\alpha + \beta)$ , applying the operator  $(\mathbf{L} - \mu \mathbf{I})^{\alpha-1} (\mathbf{L} - \nu \mathbf{I})^{\beta-1}$  to both members of (31) the assertion follows. □

It is immediate to ascertain that the above Lemma implies  $[\mathcal{D}_\mu, \mathcal{D}_\nu] \subset \mathcal{D}_\mu \oplus \mathcal{D}_\nu$ , so that the following result holds

**Proposition 21.** *Let  $\mathbf{L}$  be a Haantjes operator, and  $\mathcal{D}_\mu, \mathcal{D}_\nu$  be two distributions with Riesz indices  $\rho_\mu$  and  $\rho_\nu$ , respectively. Then, the distribution*

$$\mathcal{D}_\mu \oplus \mathcal{D}_\nu \equiv \text{Ker}(\mathbf{L} - \mu \mathbf{I})^{\rho_\mu} \oplus \text{Ker}(\mathbf{L} - \nu \mathbf{I})^{\rho_\nu}, \quad \mu \neq \nu$$

*is integrable.*

The Haantjes Theorem 17 is an immediate consequence of Propositions 19 and 21.

In [16] and [26], the integrability of the eigendistributions of a Nijenhuis operator with generalized eigenvectors of Riesz index 2 was proved. However, the case of Haantjes operators was not considered. On the other hand, to the best of our knowledge, the proofs of the Haantjes theorem available in the literature (see for instance [23], [25]) are based on the more restrictive assumption that the Haantjes operator be diagonalizable.

Let us show in detail how to determine a coordinate system that, under the assumption of Theorem 17, provides a Jordan form for  $\mathbf{L}$ . Denote by

$$(37) \quad E_i := \text{Im}(\mathbf{L} - l_i \mathbf{I})^{\rho_i} = \bigoplus_{j=1, j \neq i}^s \mathcal{D}_j, \quad i = 1, \dots, s$$

the  $(n - r_i)$ -dimensional distribution spanned by all the generalized eigenvectors of  $\mathbf{L}$ , except those associated with the eigenvalue  $l_i$ . Such a distribution will be called a *characteristic distribution* of  $\mathbf{L}$ . Let  $E_i^\circ$  denote the annihilator of the distribution  $E_i$ . Since  $\mathbf{L}$  has real eigenvalues by assumption, the tangent and cotangent spaces of  $M$  can be locally decomposed as

$$(38) \quad TM = \bigoplus_{i=1}^s \mathcal{D}_i, \quad T^*M = \bigoplus_{i=1}^s E_i^\circ.$$

Moreover, each characteristic distribution  $E_i$  is integrable by virtue of Theorem 17. We shall denote by  $\mathcal{E}_i$  the associated foliation and by  $\mathcal{S}_i(\mathbf{x})$  the connected leave through  $\mathbf{x}$  belonging to  $\mathcal{E}_i$ . Thus, the set  $(E_1, E_2, \dots, E_s)$  generates as many foliations  $(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s)$  as the eigenvalues of  $\mathbf{L}$ . Such a set of foliations will be referred to as the *characteristic web* of  $\mathbf{L}$  and the leaves  $\mathcal{S}_i(\mathbf{x})$  of each foliation  $\mathcal{E}_i$  as the *characteristic fibers* of the web.

**Definition 22.** A collection of  $r_i$  smooth functions will be said to be adapted to a foliation  $\mathcal{E}_i$  of the characteristic web of  $\mathbf{L}$  if the level sets of such functions coincide with the characteristic fibers of the foliation.

**Definition 23.** A parametrization of the characteristic web of an operator  $\mathbf{L}$  is an ordered set of  $n$  independent smooth functions  $(f_1, \dots, f_n)$  such that each ordered subset  $(f_{i_1}, \dots, f_{i_r})$  is adapted to the  $i$ -th characteristic foliation of the web:

$$(39) \quad f_{i_k|\mathcal{S}_i(\mathbf{x})} = \text{const} \quad \forall \mathcal{S}_i(\mathbf{x}) \in \mathcal{E}_i, \quad k = 1, \dots, r, \quad i_r = i_1 + r_i.$$

In this case, we shall say that the collection of functions is adapted to the web and that each of them is a characteristic function.

**Corollary 24.** Assume that  $\mathbf{L}$  has real eigenvalues. Then, the vanishing of the Haantjes tensor of  $\mathbf{L}$  is sufficient to assure that it admits an equivalence class of holonomic generalized eigenframes. Furthermore, if  $\mathbf{L}$  is semisimple the vanishing of the Haantjes tensor is also a necessary condition. In addition, if  $\mathbf{L}$  is simple each eigenframe is holonomic.

*Proof.* Since each characteristic distribution  $E_i$  is integrable by virtue of Haantjes's Theorem 17, in the corresponding annihilator  $E_i^\circ$  one can find  $r_i$  exact one-forms  $(dx_{i_1}, \dots, dx_{i_r})$  that provide functions  $(x_{i_1}, \dots, x_{i_r})$  adapted to the characteristic foliation  $\mathcal{E}_i$ . Collecting together all these functions, one can construct a set of  $n$  independent coordinates  $(x_1, \dots, x_n)$  adapted to the characteristic web.

The natural frame  $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$  turns out to be an eigenframe. In fact, as

$$(40) \quad \mathcal{D}_i^\circ = \bigoplus_{j=1, j \neq i}^s E_j^\circ,$$

the components of any generalized eigenvector  $W \in \mathcal{D}_i$  satisfy the conditions

$$(41) \quad W^j = W(x_j) = 0, \quad j \neq i_1, \dots, i_r.$$

Thus, we have that  $W = \sum_{k=1}^r W(x_{i_k}) \frac{\partial}{\partial x_{i_k}}$ , therefore

$$(42) \quad \mathcal{D}_i = \text{Span}\left\{\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_r}}\right\},$$

and each frame *equivalent* to  $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$  is a holonomic eigenframe.

Moreover, if  $\mathbf{L}$  is semisimple Prop. 7 holds true and each eigenframe fulfills the conditions of Prop. 14.  $\square$

A local chart adapted to the characteristic web of  $\mathbf{L}$  can be computed by using the transposed operator  $\mathbf{L}^T$ . Let us denote with

$$(43) \quad \text{Ker}\left(\mathbf{L}^T - l_i(\mathbf{x})\mathbf{I}\right)^{\rho_i}$$

the  $i$ -th distribution of the generalized eigen 1-forms with eigenvalue  $l_i(\mathbf{x})$ , which fulfills the property

$$(44) \quad \text{Ker}(\mathbf{L}^T - l_i\mathbf{I})^{\rho_i} = \left(\text{Im}(\mathbf{L} - l_i\mathbf{I})^{\rho_i}\right)^\circ = E_i^\circ$$

Such a property implies that each generalized eigenform of  $\mathbf{L}^T$  annihilates all generalized eigenvectors of  $\mathbf{L}$  with different eigenvalues. Moreover, it allows to prove that

**Proposition 25.** *Let  $\mathbf{L}$  be a Haantjes operator. The differentials of the characteristic coordinate functions are exact generalized eigenforms for the transposed operator  $\mathbf{L}^T$ . Conversely, each (locally) exact generalized eigenform of  $\mathbf{L}^T$  provides a characteristic function for the Haantjes web of  $\mathbf{L}$ .*

The characteristic functions of a Haantjes operator are characterized by the following simple property.

**Proposition 26.** *A function  $h$  on  $M$  is a characteristic function of a Haantjes operator associated with the eigenvalue  $l_i$  if and only if, given a set of local coordinates adapted to the characteristic web  $(x_1, \dots, x_n)$ ,  $h$  depends, at most, on the subset of coordinates  $(x_{i_1}, \dots, x_{i_r})$  that are constant over the leaves of the foliation  $\mathcal{E}_i$ .*

*Proof.* If  $h = h(x_{i_1}, \dots, x_{i_r})$ , it is constant on the leaves of  $\mathcal{E}_i$ , then  $dh \in E_i^\circ$ . Viceversa, if we assume that  $dh \in E_i^\circ$ , then it can be expressed in terms of a linear combination of  $\{dx_{i_1}, \dots, dx_{i_r}\}$  only. The thesis follows from the exactness of  $dh$ .  $\square$

**Remark 27.** *In particular, if  $\mathbf{L}$  is a semisimple Nijenhuis operator, its eigenvalues are characteristic functions for the Haantjes web, according to Proposition 9.*

**Remark 28.** [26] *Let us suppose that a generic operator  $\mathbf{L}$  admits a symmetry, i.e. a vector field  $X$  such that*

$$(45) \quad \mathcal{L}_X(\mathbf{L}) = 0.$$

*In this case, the operator  $\mathbf{L}$  will be called a recursion operator for  $X$ . Then, the eigenvalues of  $\mathbf{L}$  as well are invariant along the flow of  $X$  and the corresponding generalized eigen-distributions are stable, i.e.*

$$(46) \quad \mathcal{L}_X(l_i) = 0, \quad \mathcal{L}_X(\mathcal{D}_i) \subseteq \mathcal{D}_i, \quad \mathcal{L}_X(E_i^\circ) \subseteq E_i^\circ \quad \forall i = 1, 2, \dots, s.$$

## 4. THE THEORY OF SYMPLECTIC-HAANTJES MANIFOLDS

In this section, we shall introduce the new class of symplectic-Haantjes manifolds, that we shall call the  $\omega\mathcal{H}$  manifolds by analogy with the known  $\omega N$  ones [43]. The main reason to define these manifolds is that they provide a natural setting in which the theory of Hamiltonian integrable systems can be properly formulated.

**Definition 29.** A *symplectic-Haantjes* or  $\omega\mathcal{H}$  manifold  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  is a symplectic manifold of dimension  $2n$ , endowed with  $n$  endomorphisms of the tangent bundle of  $M$

$$\mathbf{K}_\alpha : TM \mapsto TM \quad \alpha = 0, \dots, n-1 ,$$

which satisfy the following conditions:

- $\mathbf{K}_0 = \mathbf{I}$ .
- Their Haantjes tensor vanishes identically, i.e.

$$(47) \quad \mathcal{H}_{\mathbf{K}_\alpha}(X, Y) = 0 \quad \forall X, Y \in TM, \quad \alpha = 0, \dots, n-1 .$$

- The endomorphisms are compatible with  $\omega$  (or equivalently, with the corresponding symplectic operator  $\mathbf{\Omega} := \omega^\flat$ ):

$$(48) \quad \mathbf{K}_\alpha^T \mathbf{\Omega} = \mathbf{\Omega} \mathbf{K}_\alpha, \quad \alpha = 0, \dots, n-1 ,$$

that is, the operators  $\mathbf{\Omega} \mathbf{K}_\alpha$  are skew symmetric.

- The endomorphisms are compatible with each others, namely they form a commutative ring  $\mathcal{K}$

$$(49) \quad \mathbf{K}_\alpha \mathbf{K}_\beta = \mathbf{K}_\beta \mathbf{K}_\alpha, \quad \alpha, \beta = 0, \dots, n-1 ,$$

and also generate a module over the ring of smooth functions on  $M$ :

$$(50) \quad \mathcal{H}_{\left(\sum_{\alpha=0}^{n-1} a_\alpha(\mathbf{x}) \mathbf{K}_\alpha\right)}(X, Y) = 0 , \quad \forall X, Y \in TM ,$$

where  $a_\alpha(\mathbf{x})$  are arbitrary smooth functions on  $M$ .

The  $(n+1)$ -ple  $(\omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  will be called the  $\omega\mathcal{H}$  structure associated with the  $\omega\mathcal{H}$  manifold and  $\mathcal{K}$  the Haantjes module (ring).

The above conditions can be re-cast by saying that, essentially, the endomorphisms  $\mathbf{K}_\alpha$  are Haantjes operators, compatible both with  $\omega$  and with each others. Moreover, the assumption (49) ( respectively (50)) assures that each operator belonging to the ring (respectively the module)  $\mathcal{K}$  is a Haantjes operator compatible with  $\omega$  and with the original Haantjes operators  $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$ .

As a consequence of the above conditions, we get the following simple proposition that turns out to be crucial for many results of the present theory.

**Proposition 30.** Given a  $\omega\mathcal{H}$  structure, let  $\mathbf{P} = \mathbf{\Omega}^{-1}$  be the Poisson operator associated with the symplectic structure  $\omega$ . Any composed operator  $\mathbf{\Omega} \mathbf{K}_\alpha$ ,  $\mathbf{P} \mathbf{K}_\beta^T$ ,  $\mathbf{K}_\beta^T \mathbf{\Omega} \mathbf{K}_\alpha$ ,  $\mathbf{K}_\alpha \mathbf{P} \mathbf{K}_\beta^T$ ,  $\mathbf{\Omega}(\mathbf{K}_\alpha - f(\mathbf{x})\mathbf{I})^m$ ,  $\alpha, \beta = 0, \dots, n-1$ ,  $m \in \mathbb{N}$  is skew symmetric.

For instance, it has important consequences on the spectrum of the Haantjes operators.

**Corollary 31.** Given a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold  $M$ , we will suppose that the number of distinct eigenvalues of each operator  $\mathbf{K}_\alpha$ , as well as the dimension of the related eigenspaces, do not depend on the point  $\mathbf{x}$ , at least for  $\mathbf{x}$  in a dense open

subset of  $M$ . Then, every generalized eigen-distribution  $\text{Ker}(\mathbf{K}_\alpha - l_i^{(\alpha)} \mathbf{I})^{m_i^{(\alpha)}}$ ,  $m_i^{(\alpha)} = 1, \dots, \rho_i^{(\alpha)}$ , is even-dimensional. Therefore each eigenvalue  $l_i^{(\alpha)}(\mathbf{x})$  has both its geometric multiplicity ( $\dim(\text{Ker}(\mathbf{K}_\alpha - l_i^{(\alpha)} \mathbf{I}))$ ) and its algebraic multiplicity ( $\dim(\text{Ker}(\mathbf{K}_\alpha - l_i^{(\alpha)} \mathbf{I})^{\rho_i^{(\alpha)}})$ ) even.

*Proof.* In a  $\omega\mathcal{H}$  manifold every generalized eigen-distribution  $\text{Ker}(\mathbf{K}_\alpha - l_i^{(\alpha)} \mathbf{I})^{m_i}$  has the same dimension of the kernel of the operator  $\Omega(\mathbf{K}_\alpha - l_i^{(\alpha)} \mathbf{I})^{m_i}$ , which is skew-symmetric by virtue of Proposition 30.  $\square$

Due to the above corollary, the number of the eigenvalues of a Haantjes operator  $\mathbf{K}_\alpha$  is less or equal to  $n$ , therefore their algebraic multiplicity is at least 2.

**Definition 32.** *If the number of distinct eigenvalues of a Haantjes operator  $\mathbf{K} \in \mathcal{K}$  is exactly  $n$ , we say that such an operator is maximal.*

Let us denote with  $m_{\mathbf{K}}(\lambda)$  the minimal polynomial of  $\mathbf{K}$ .

**Lemma 33.** *A Haantjes operator of a  $\omega\mathcal{H}$  structure is maximal if and only if its minimal polynomial is the product of  $n$  linear factors*

$$(51) \quad m_{\mathbf{K}}(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) .$$

We shall also consider a particular class of  $\omega\mathcal{H}$  manifold, especially relevant for the applications.

**Definition 34.** *A  $\omega\mathcal{H}$  manifold  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  endowed with a vector field  $X \in TM$  such that*

$$(52) \quad \mathcal{L}_X(\omega) = 0, \quad \mathcal{L}_X(\mathbf{K}_\alpha) = 0, \quad \alpha \in \{1 \dots, n-1\} ,$$

*will be called a symplectic-Haantjes manifold with a (locally) Hamiltonian symmetry vector field  $X$ .*

According to Remark 28, each operator  $\mathbf{K}_\alpha$  is a recursion operator for the Hamiltonian symmetry  $X$  and their eigenvalues are integrals of motion for  $X$ .

**4.1. Lenard-Haantjes chains.** The theory of Lenard chains is a fundamental piece of the geometric approach to soliton hierarchies. Lenard chains have been introduced in order to construct integrals of motion in involution for infinite-dimensional Hamiltonian systems [34, 35] (see also [56], for a brief history about the origin of the name “Lenard chains”). However, only recently some non trivial generalizations of Lenard chains have proved to be useful in the study of separation of variables for finite-dimensional Hamiltonian systems (see [48, 49, 62, 17, 19] and reference therein).

Hereafter, as a byproduct of our approach à la Haantjes, we propose a further generalization of the standard notions of the theory, which has the advantage to be both simple and directly connected to the theory of classical integrable systems.

**Definition 35.** *Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold and  $\{H_j\}_{1 \leq j \leq n}$  be  $n$  independent functions which satisfy the following relations*

$$(53) \quad dH_j = \tilde{\mathbf{K}}_\alpha^T dH , \quad j = \alpha + 1, \quad \alpha = 0, \dots, n-1, \quad H := H_1,$$

where

$$(54) \quad \tilde{\mathbf{K}}_\alpha := \sum_{\beta=0}^{n-1} a_{ij}(\mathbf{x}) \mathbf{K}_\beta, \quad i = \alpha + 1, \quad j = \beta + 1, \quad \alpha, \beta = 0, \dots, n-1,$$

and  $a_{ij}(\mathbf{x})$  are suitable smooth functions on  $M$ , that are assumed to satisfy the properties

- i)  $a_{1k} = \delta_{1k}$ ,  $k = 1, 2, \dots, n$ ;
- ii)  $a_{jk}$  are elements of an invertible matrix-valued function  $\mathbf{A}$ , called the transition matrix between  $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$  and  $\{\tilde{\mathbf{K}}_0, \tilde{\mathbf{K}}_1, \dots, \tilde{\mathbf{K}}_{n-1}\}$ .

Under these conditions, we shall say that the functions  $\{H_j\}_{1 \leq j \leq n}$  form a Lenard-Haantjes chain generated by the function  $H$ .

**Remark 36.** The operators  $\tilde{\mathbf{K}}_\alpha$  belong to the Haantjes module  $\mathcal{K}$  generated by the operators  $\mathbf{K}_\alpha$ , so they are also Haantjes operators as a consequence of the assumption (50). Moreover, they are compatible with  $\omega$  and commute with each other. Consequently, the operators  $\tilde{\mathbf{K}}_\alpha$  endow  $M$  with the  $\omega\mathcal{H}$  structure  $(M, \omega, \tilde{\mathbf{K}}_0, \tilde{\mathbf{K}}_1, \dots, \tilde{\mathbf{K}}_{n-1})$ , adapted to the function  $H$ , that we shall call a modified  $\omega\mathcal{H}$  structure.

To enquire about the existence of Lenard-Haantjes chains, we have to consider the co-distribution  $\mathcal{D}_H^\circ$  generated by  $H$  through the (transposed of) the Haantjes operators  $\mathbf{K}_\alpha$

$$(55) \quad \mathcal{D}_H^\circ := \text{Span}\{dH, \mathbf{K}_1^T dH, \dots, \mathbf{K}_{n-1}^T dH\},$$

and to compare the distribution  $\mathcal{D}_H$  of the vector fields annihilated by  $\mathcal{D}_H^\circ$ , with the distribution of the vector fields symplectically orthogonal to  $\mathcal{D}_H$ . We shall denote it by  $\mathcal{D}_H^\perp$ ; it can be represented as

$$(56) \quad \mathcal{D}_H^\perp = \mathbf{P}(\mathcal{D}_H^\circ) = \text{Span}\{X_H, \mathbf{K}_1 X_H, \mathbf{K}_2 X_H, \dots, \mathbf{K}_{n-1} X_H\},$$

where  $X_H = \mathbf{P} dH$  is the Hamiltonian vector field with Hamiltonian function  $H$ . Indeed, from Proposition 30 we deduce the following result.

**Proposition 37.** Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold, and  $H$  be a smooth function on  $M$ . The relation

$$(57) \quad \mathcal{D}_H^\perp \subseteq \mathcal{D}_H$$

holds true. Therefore,  $\mathcal{D}_H$  is a co-isotropic distribution and  $\mathcal{D}_H^\perp$  is an isotropic one. Moreover, if

$$\dim(\mathcal{D}_H(\mathbf{x})) = n \quad \forall \mathbf{x} \in M$$

they coincide and form a Lagrangian distribution.

*Proof.* Each vector field belonging to  $\mathcal{D}_H^\perp$  is annihilated by the one-form belonging to  $\mathcal{D}_H^\circ$  as

$$\langle \mathbf{K}_\alpha^T dH, \mathbf{K}_\beta X_H \rangle = \langle dH, \mathbf{K}_\alpha \mathbf{K}_\beta \mathbf{P} dH \rangle \stackrel{(30)}{=} 0$$

□

The following theorem states the necessary and sufficient conditions to ensure the existence of a Lenard-Haantjes chain generated by a function  $H$ .



**Theorem 38.** *Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold, and  $H$  be a smooth function on  $M$ . Let  $\mathcal{D}_H^\circ$  be the co-distribution spanned by the set of 1-forms*

$$(58) \quad \beta_{\alpha+1} := \mathbf{K}_\alpha^T dH \quad \alpha = 0, \dots, n-1,$$

*that we assume to be of rank  $n$  (independent on  $\mathbf{x}$ ) and let  $\mathcal{D}_H$  be the distribution of the vector fields annihilated by them. Then, the function  $H$  generates a Lenard-Haantjes chain (53) if and only if  $\mathcal{D}_H^\circ$  ( $\mathcal{D}_H$ ) is integrable in the sense of Frobenius. Under this assumption, the foliation  $\mathcal{F}_H$  of  $\mathcal{D}_H^\circ$  is a Lagrangian foliation.*

*Proof.* By definition, the Lenard-Haantjes chain (53) contains  $n$  exact 1-forms. Therefore they generate the integrable Lagrangian distribution

$$(59) \quad \mathcal{D}^\circ = \text{Span}\{dH_1, \dots, dH_n\}$$

which coincides with  $\mathcal{D}_H^\circ$ , by virtue of the linear relation (54) among the  $\mathbf{K}_\alpha$  and the  $\tilde{\mathbf{K}}_\alpha$ .

Viceversa, let  $\mathcal{D}_H$  be integrable and  $\mathcal{F}_H$  be the associated foliation. Then, there exist  $n$  independent functions  $(H_1, H_2, \dots, H_n)$  which are constant on the leaves of  $\mathcal{F}_H$ . Their differentials belong to  $\mathcal{D}_H^\circ$ , hence

$$(60) \quad dH_i = \sum_{j=1}^n a_{ij}(\mathbf{x}) \beta_j \stackrel{(58)}{=} \left( \sum_{j=1}^n a_{ij}(\mathbf{x}) \mathbf{K}_{j-1}^T \right) dH \quad j = 1, \dots, n.$$

The choice  $H_1 = H$  implies that  $\tilde{\mathbf{K}}_0 = \mathbf{I}$  and, consequently, property **i)** in the Definition 35. The fact that  $\{\beta_1, \dots, \beta_n\}$  and  $\{dH_1, \dots, dH_n\}$  pointwise are two different basis of  $\mathcal{D}_H^\circ(\mathbf{x})$ , implies the property **ii)**. In this manner, we can construct the modified  $\omega\mathcal{H}$  structure (54), equivalent to the original one but adapted to the Lenard-Haantjes chain (53) generated by the function  $H$ . Furthermore, the functions  $(H_1, H_2, \dots, H_n)$  are in involution by virtue of Theorem 41 below, therefore they generate a *Lagrangian* foliation w.r.t. the symplectic form  $\omega$ . Such a foliation coincides with the foliation  $\mathcal{F}$  to which the Hamiltonian vector fields  $(X_{H_1}, X_{H_2}, \dots, X_{H_n})$  are tangent.  $\square$

**Corollary 39.** *Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold, and  $H$  be the Hamiltonian function of a Hamiltonian vector field integrable in the sense of Liouville. Also, let  $(H_1, H_2, \dots, H_n)$  be a set of independent integrals in involution and  $\mathcal{D}^\circ$  the co-distribution spanned by their differentials. Such integrals form a Lenard-Haantjes chain generated by  $H = H_1$  if and only if  $H$  satisfies the following conditions*

$$(61) \quad \mathbf{K}_\alpha^T dH \in \mathcal{D}^\circ, \quad \alpha = 0, 1, \dots, n-1.$$

*Proof.* Conditions (61) are equivalent to say that  $\mathcal{D}_H^\circ \subseteq \mathcal{D}^\circ$ . Therefore,  $\mathcal{D}_H^\circ \equiv \mathcal{D}^\circ$  as they both have, by assumption, the same rank  $n$ . Thus,  $\mathcal{D}_H$  is integrable and, by virtue of the previous theorem, it follows that the function  $H$  is the generator of the Lenard-Haantjes chain formed by  $(H_1, H_2, \dots, H_n)$ .

Conversely, if  $(H_1, H_2, \dots, H_n)$  form a Lenard-Haantjes chain generated by  $H$ , by Definition (53) it follows that  $\tilde{\mathbf{K}}_\alpha^T dH \in \mathcal{D}^\circ$ , for  $\alpha = 0, 1, \dots, n-1$ . Then, since formula (54) is invertible by hypothesis, conditions (61) are fulfilled.  $\square$

**4.2. Symplectic-Nijenhuis manifolds and Lenard-Nijenhuis chains.** A particular but especially relevant class of  $\omega\mathcal{H}$  structures is represented by the  $\omega N$  manifolds [43, 36]. They are symplectic manifolds endowed with a *single* endomorphism of the tangent bundle,  $N : TM \mapsto TM$  that satisfies the following conditions:

- its Nijenhuis torsion (4) vanishes identically, i.e.  $\forall X, Y \in TM$

$$(62) \quad \mathcal{T}_N(X, Y) = 0;$$

- it is compatible with  $\omega$ , that is, the tensor  $P_1 = N\Omega^{-1}$  is also a Poisson tensor and is compatible with  $P_0 := \Omega^{-1}$ , i.e. the Schouten bracket of  $P_0$  and  $P_1$  vanishes.

The above conditions, that amount to say that  $N$  is a Nijenhuis (or hereditary) operator compatible with  $\omega$ , in turn ensure that the  $\omega N$  structures represent a special class of *bi-Hamiltonian structures*.

**Remark 40.** *The requirement that the Nijenhuis torsion of  $N$  vanishes implies that the Haantjes tensor (6) of  $N$  vanishes as well. Thus Nijenhuis operators are a special class of Haantjes operators.*

Then, given a  $\omega N$  manifold, one can construct directly a  $\omega\mathcal{H}$  structure by choosing as Haantjes operators the first  $(n - 1)$  powers of  $N$

$$(63) \quad K_\alpha \equiv N^\alpha, \quad \alpha = 0, 1, \dots, n - 1,$$

provided that they are linearly independent. It is easy to prove that the Haantjes operators so constructed are compatible with  $\omega$ , due to the algebraic compatibility between  $N$  and  $\omega$ . Moreover, they are compatible with each other since they are powers of the same operator.

In a  $\omega N$  manifold one can construct a special class of Lenard-Haantjes chains. In fact, in this context Theorem 38 amounts to say that the co-distribution

$$(64) \quad \mathcal{D}_H^\circ = \text{Span}\{dH, N^T dH, \dots, (N^T)^{n-1} dH\}$$

which is supposed to be of rank  $n$ , is integrable if and only if  $H$  generates the following Lenard-Haantjes chain

$$(65) \quad dH_j = \tilde{K}_\alpha dH = p_\alpha(N^T) dH,$$

with

$$(66) \quad p_\alpha(N) = \sum_{k=1}^n a_{jk}(\mathbf{x}) N^{k-1}, \quad \alpha = 0, 1, \dots, n - 1, \quad j = \alpha + 1.$$

Here  $a_{jk}(\mathbf{x})$  are suitable coefficients fulfilling the requirements of Definition 35. As the modified Haantjes operators  $\tilde{K}_\alpha$  are generated by a unique Nijenhuis operator  $N$ , the chain (65) will be called a Lenard-Nijenhuis chain. Note that this type of chains have been called Nijenhuis chains in [17], and generalized Lenard chains in [58, 61]. The particular chains with  $\tilde{K}_\alpha \equiv K_\alpha = N^\alpha$  are the classical Lenard-Magri chains.

## 5. COMPLETE INTEGRABILITY AND HAANTJES STRUCTURES

The aim of this Section is to prove one of the main results of this paper. Also, we shall show in a specific example how the Haantjes formulation overcomes, for the vector field under scrutiny, an obstruction to the existence of a classical Lenard chain pointed out by Brouzet.

**5.1. Haantjes theorem for integrable systems.** We propose a characterization of the notion of integrability in the sense of Liouville–Arnold in terms of  $\omega\mathcal{H}$  structures.

**Theorem 41.** *Let  $M$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold and  $\{H_1, H_2, \dots, H_n\}$  be smooth functions forming a Lenard-Haantjes chain. Then, the foliation generated by these functions is Lagrangian. Consequently, each Hamiltonian system, with Hamiltonian functions  $H_j$ ,  $1 \leq j \leq n$  is integrable by quadratures.*

*Conversely, let  $\{H_1, \dots, H_n\}$  be a completely integrable system in  $n$  dimensions, defined by a Hamiltonian  $H = H_1$  and a set of independent integrals of motion  $H_2, \dots, H_n$ . Let  $\{(J_k, \phi_k)\}$ ,  $k = 1, \dots, n$ , denote a set of action-angle variables, with associated frequencies  $\nu_k := \frac{\partial H}{\partial J_k}$ . If  $H$  is non degenerate [3], that is*

$$(67) \quad \det \left( \frac{\partial \nu_k}{\partial J_i} \right) = \det \left( \frac{\partial^2 H}{\partial J_i \partial J_k} \right) \neq 0 ,$$

*then  $M$  admits, in any tubular neighborhood of an Arnold torus, an  $\omega\mathcal{H}$  structure given by*

$$(68) \quad \mathbf{K}_\alpha = \sum_{i=1}^n \frac{\nu_i^{(\alpha+1)}}{\nu_i} \left( \frac{\partial}{\partial J_i} \otimes dJ_i + \frac{\partial}{\partial \phi_i} \otimes d\phi_i \right) \quad \alpha = 0, \dots, n-1 ,$$

*where  $\nu_i^{(\alpha+1)}$  are the frequencies of the  $(\alpha+1)$  linear flow.*

*Proof.* By virtue of the classical Arnold-Liouville theorem, it is sufficient to prove that the functions  $H_j$  belonging to a Lenard-Haantjes chain are in involution w.r.t. the Poisson bracket defined by the symplectic form  $\omega$ . In fact,

$$(69) \quad \{H_j, H_k\} = \langle dH_j, \mathbf{P} dH_k \rangle = \langle \tilde{\mathbf{K}}_\alpha^T dH, \mathbf{P} \tilde{\mathbf{K}}_\beta^T dH \rangle = \langle dH, \tilde{\mathbf{K}}_\alpha \mathbf{P} \tilde{\mathbf{K}}_\beta^T dH \rangle \stackrel{Prop. 30}{=} 0$$

Let us prove the converse statement. The integrals of motion  $\{H_1, \dots, H_n\}$  are all assumed to be smooth functions on an open dense subset of the phase space and in involution among each others. Due to the celebrated Arnold theorem [3], the  $2n$ -dimensional phase space is foliated by leaves whose connected components, if compact, are invariant tori. Also, there exists at least in any tubular neighborhood of each torus a set of action-angle (AA) variables  $\{(J_i, \phi_i)\}$ , in which the symplectic 2-form reads

$$(70) \quad \omega = \sum_{i=1}^n dJ_i \wedge d\phi_i .$$

Owing to condition (67), the set  $\{H_1, \dots, H_n\}$  depends on the action variables only. Then the functions  $H_i$  take the generic form

$$(71) \quad H_i = H_i(J_1, \dots, J_n), \quad i = 1, \dots, n.$$

With these data, we shall construct a semi-global  $\omega\mathcal{H}$  structure associated with  $\mathcal{S}$ .

We can take as Haantjes operators the following *diagonal* operators in the action-angle coordinates

$$(72) \quad \mathbf{K}_\alpha = \sum_{i=1}^n l_i^{(\alpha)} \frac{\partial}{\partial J_i} \otimes dJ_i + \sum_{i=1}^n l_{n+i}^{(\alpha)} \frac{\partial}{\partial \phi_i} \otimes d\phi_i \quad \alpha = 0, \dots, n-1 ,$$

so that, at the same time, they have their Haantjes tensor vanishing, as required in eq. (47), they commute with each others, according to eq. (49), and fulfill the differential compatibility condition (50).

Once we have chosen the natural vector fields  $(\frac{\partial}{\partial J_i}, \frac{\partial}{\partial \phi_i})$  as a basis of shared eigenvectors for the Haantjes operators, it remains to determine their eigenvalues  $(l_i^{(\alpha)}, l_{n+i}^{(\alpha)})$ .

Notice that the algebraic compatibility conditions (48) of the operators (72) with the symplectic form (70) are equivalent to the conditions

$$(73) \quad l_{n+i}^{(\alpha)} = l_i^{(\alpha)} \quad i = 1, \dots, n \quad \alpha = 1, \dots, n-1.$$

Thus, the eigenvalues of our Haantjes operators (72) must be at least double. Finally, we impose that the integrals of motion  $(H_1, H_2, \dots, H_n)$  form a Lenard Haantjes chain generated by  $H = H_1$ , i.e.

$$(74) \quad \mathbf{K}_\alpha^T dH = dH_{\alpha+1}, \quad \alpha = 0, \dots, n-1.$$

Being  $\mathbf{K}_\alpha$  diagonal in the AA variables, such conditions are equivalent to the following system of  $2n$  algebraic equations in the  $n$  indeterminate functions  $l_i^{(\alpha)}$ ,

$$(75) \quad l_i^{(\alpha)} \frac{\partial H}{\partial J_i} = \frac{\partial H_{\alpha+1}}{\partial J_i}, \quad i = 1, \dots, n.$$

$$(76) \quad l_i^{(\alpha)} \frac{\partial H}{\partial \phi_i} = \frac{\partial H_{\alpha+1}}{\partial \phi_i}$$

Obviously, eqs. (76) are trivially satisfied, so that only eqs. (75) should be taken into account. Assuming, without loss of generality, that  $\nu_i \neq 0$ ,  $i = 1, \dots, n$ , the equations (75) imply that the elements in the diagonal of the  $k$ -th operators must be the ratios between the corresponding frequencies associated to the  $k$ -th integral and to the Hamiltonian, respectively. It is easy to prove that the Haantjes operators so obtained are independent, due to the independence of the integrals of motion. Consequently, the Haantjes operators that provide a Lenard-Haantjes chain formed by  $\{H_1, \dots, H_n\}$  read as in eq. (68).  $\square$

**Remark 42.** *The Haantjes operators (68) exist without any restriction on the form of the Hamiltonian functions (71), except for the condition (67). However, if one wishes to construct a Nijenhuis recursion operator  $\mathbf{N}$  for  $H$ , i.e. a Nijenhuis operator that, at the same time, provides a classical Lenard chain*

$$(77) \quad dH_j = (\mathbf{N}^T)^{(j-1)} dH,$$

*and has the natural vector fields  $(\frac{\partial}{\partial J_i}, \frac{\partial}{\partial \phi_i})$  as eigenvectors, then the Hamiltonian function must be necessarily of the separated form*

$$(78) \quad H(J_1, J_2, \dots, J_n) = \sum_{i=1}^n H_i(J_i),$$

*where  $H_i(J_i)$  is a smooth function of the single action variable  $J_i$  (see [43], [46]).*

**Remark 43.** *The eigenvalues  $l_i^{(\alpha)}$  of the Haantjes operators  $\mathbf{K}_\alpha$ ,  $\alpha = 1, 2, \dots, n-1$ , depending only on action variables, are integrals of motion for the Hamiltonian vector field  $X_H$ , i.e. their Lie derivatives along the flow of  $X_H$  vanishes:*

$$\mathcal{L}_{X_H} l_i^{(\alpha)} = 0.$$

However, this property does not imply that the Haantjes operators are recursion operators for  $X_H$  as

$$(79) \quad \mathcal{L}_{X_H} K_\alpha = \sum_{i,k=1}^n (l_i^{(\alpha)} - l_k^{(\alpha)}) \frac{\partial \nu_i}{\partial J_k} \frac{\partial}{\partial \phi_i} \otimes dJ_k .$$

**5.2. The analysis of Brouzet.** In [12], R. Brouzet studied the existence for a completely integrable system of a Nijenhuis *recursion* operator, that is a Nijenhuis operator *compatible* with  $\omega$  and fulfilling eq. (45). He proved that the existence of a Nijenhuis recursion operator for  $X_H$  requires very strong conditions on the form of its Hamiltonian function. Accordingly, he presented an example of an integrable system with two degrees of freedom that does not admit a recursion operator compatible with the original symplectic structure. Here we show that such example does admit a simple formulation in the context of the  $\omega\mathcal{H}$  geometry.

In his analysis, Brouzet considered the symplectic manifold  $M = \mathbb{R}^2 \times \mathbb{T}^2$ , with the action variables  $(J_1, J_2) \in \mathbb{R}^2$ , the angles  $(\phi_1, \phi_2)$  on the bi-dimensional torus  $\mathbb{T}^2$ , and the Hamiltonian function

$$(80) \quad H = J_1(1 + J_2^2) ,$$

which is not of the form (78) and is non degenerate according to (67), in the dense open submanifold  $M' := \{m \in M : J_2 \neq 0\}$ . The corresponding Hamiltonian vector field

$$(81) \quad X_H = (1 + J_2^2) \frac{\partial}{\partial \phi_1} + 2J_1 J_2 \frac{\partial}{\partial \phi_2}$$

is completely integrable, since any smooth function depending only on the actions is an integral of motion for it. For instance, let us take

$$(82) \quad H_2 = J_2^2 ,$$

which is functionally independent of  $H$  in  $M'$ . One can easily verify that the two Hamiltonian functions in involution  $(H_1 = H, H_2)$  form a Lenard-Haantjes chain w.r.t. the  $\omega\mathcal{H}$  structure given by the standard symplectic form

$$(83) \quad \omega = dJ_1 \wedge d\phi_1 + dJ_2 \wedge d\phi_2$$

and by the Haantjes operators

$$(84) \quad K_0 = I , \quad K_1 = \frac{1}{J_1} \left( \frac{\partial}{\partial J_2} \otimes dJ_2 + \frac{\partial}{\partial \phi_2} \otimes d\phi_2 \right) ,$$

constructed in the open submanifold of  $M'$  where  $J_1 \neq 0$ , according to the prescriptions (68).

It is interesting to observe that the authors of [32] have by-passed the Brouzet obstruction to the definition of a Nijenhuis recursion operator for the Hamiltonian (80) (and for other examples presented in [13]) by using a different strategy. The alternative approach consists in allowing a Nijenhuis recursion operator compatible with a symplectic structure different from the original one. By contrast, in our theory, the Haantjes operators are compatible with the very original symplectic structure.

## 6. NEW INTEGRABLE MODELS FROM HAANTJES GEOMETRY

Once we have stated the conceptual equivalence between complete integrability of a Hamiltonian system and the existence of an associated Haantjes structure, we can use this equivalence in both ways: to construct integrable models from a given Haantjes geometry (the *direct problem*) or conversely to determine the Haantjes geometry of a given integrable system (the *inverse problem*). In this section, we will adopt the first point of view, in order to show the flexibility of the Haantjes approach in applicative contexts. Indeed, by imposing the existence of a Lenard-Haantjes chain generated by a specific Haantjes operator, we define classes of associated integrable models.

**6.1. Harmonic functions and integrable systems.** Even the case of uniform Haantjes operators (i.e. independent of  $\mathbf{x} \in M$ ) provides an interesting class of integrable models directly related to the theory of analytic functions.

**Theorem 44.** *Let  $M$  be a symplectic manifold of dimension 4,  $(x, y, p_x, p_y)$  a set of Darboux coordinates in  $M$ ,  $\varphi(p_x, p_y) = \varphi_1(p_x, p_y) + i\varphi_2(p_x, p_y)$  and  $\psi = \psi_1(x, y) + i\psi_2(x, y)$  be two analytic functions ( $i^2 = -1$ ). Then, the Hamiltonian*

$$(85) \quad H_1(x, y, p_x, p_y) = \varphi_1(p_x, p_y) + \psi_1(x, y)$$

*admits the first integral of motion*

$$(86) \quad H_2(x, y, p_x, p_y) = \varphi_2(p_x, p_y) + \psi_2(x, y) .$$

*Proof.* Consider the uniform Haantjes operator

$$(87) \quad \mathbf{K} = -\frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx - \frac{\partial}{\partial p_x} \otimes dp_y + \frac{\partial}{\partial p_y} \otimes dp_x .$$

We construct the Lenard-Haantjes chain

$$(88) \quad \mathbf{K}^T dH_1 = dH_2 .$$

It amounts to solve the systems of equations

$$(89) \quad \begin{cases} \frac{\partial H_1}{\partial p_y} = \frac{\partial H_2}{\partial p_x} , \\ \frac{\partial H_1}{\partial p_x} = -\frac{\partial H_2}{\partial p_y} , \end{cases}$$

and

$$(90) \quad \begin{cases} \frac{\partial H_1}{\partial y} = \frac{\partial H_2}{\partial x} , \\ \frac{\partial H_1}{\partial x} = -\frac{\partial H_2}{\partial y} , \end{cases}$$

for the functions  $\varphi_1, \varphi_2$  and  $\psi_1, \psi_2$  respectively. These relations are nothing but the classical Cauchy-Riemann equations. Hence, these functions and consequently  $H_1$  and  $H_2$  are all harmonic functions.  $\square$

**Example 45.** *A simple class of integrable models arises when both analytic functions  $\varphi(p_x, p_y)$  and  $\psi(x, y)$  are chosen to have a polynomial structure. For instance, choosing a third degree homogeneous polynomial in the momenta  $(p_x, p_y)$ , we have the system*

$$(91) \quad H_1 = p_x^3 - 3p_x p_y^2 + x^2 - y^2 , \quad H_2 = 3p_x^2 p_y - p_y^3 + 2xy .$$

**6.2. Waves and integrable systems.** Solutions of the wave equations also define integrable systems, via the Haantjes geometry.

**Theorem 46.** *Let  $\xi = \frac{x+y}{\sqrt{2}}$ ,  $\eta = \frac{x-y}{\sqrt{2}}$ ,  $p_\xi = \frac{p_x+p_y}{\sqrt{2}}$ ,  $p_\eta = \frac{p_x-p_y}{\sqrt{2}}$  characteristic coordinates and momenta in an open set of  $M$ . The Hamiltonian*

$$(92) \quad H_1(\xi, \eta, p_\xi, p_\eta) = f(\eta) + g(\xi) + F(p_\eta) + G(p_\xi)$$

where  $f, g, F, G$  are arbitrary functions of their arguments, is integrable and admits the first integral of motion

$$(93) \quad H_2(\xi, \eta, p_\xi, p_\eta) = -f(\eta) + g(\xi) - F(p_\eta) + G(p_\xi).$$

*Proof.* Consider the uniform Haantjes operator in cartesian coordinates and momenta

$$(94) \quad \mathbf{K} = \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial p_x} \otimes dp_y + \frac{\partial}{\partial p_y} \otimes dp_x.$$

We construct the Lenard-Haantjes chain

$$(95) \quad \mathbf{K}^T dH_1 = dH_2.$$

This chain is defined by the differential relations

$$(96) \quad \begin{cases} \frac{\partial H_1}{\partial p_y} = \frac{\partial H_2}{\partial p_x}, \\ \frac{\partial H_1}{\partial p_x} = \frac{\partial H_2}{\partial p_y}, \end{cases} \quad \begin{cases} \frac{\partial H_1}{\partial y} = \frac{\partial H_2}{\partial x}, \\ \frac{\partial H_1}{\partial x} = \frac{\partial H_2}{\partial y}, \end{cases}$$

These equations can be combined to provide the wave equations

$$H_{i,p_x p_x} - H_{i,p_y p_y} = 0, \quad H_{i,xx} - H_{i,yy} = 0, \quad i = 1, 2.$$

Therefore the Hamiltonian functions

$$(97) \quad H_1(x, y, p_x, p_y) = F(p_x - p_y) + G(p_x + p_y) + f(x - y) + g(x + y)$$

and

$$(98) \quad H_2(x, y, p_x, p_y) = -F(p_x - p_y) + G(p_x + p_y) - f(x - y) + g(x + y),$$

where  $F, G, f, g$  are arbitrary smooth functions of their arguments, define a completely integrable system, separable in the coordinates  $(\xi, \eta, p_\xi, p_\eta)$ .  $\square$

**Example 47.** *Choosing the functions  $F, G, f, g$  as a power of their argument, we get the interesting class of models*

$$(99) \quad H_1 = (p_x - p_y)^n + (p_x + p_y)^n + (x - y)^m + (x + y)^m,$$

$$(100) \quad H_2 = -(p_x - p_y)^n + (p_x + p_y)^n - (x - y)^m + (x + y)^m.$$

For  $n = 2$ , the Hamiltonian function  $H_1$  is quadratic in the momenta and corresponds to a class of separable systems that can be found in [54] (page 81). In particular, for  $n = 2, m = 3$  one gets the Sawada-Kotera system [1]. For  $n > 2$  one gets, to the best of our knowledge, a new family of integrable systems.

The inverse method outlined in this section can be widely adopted to generate new models from known Haantjes operators. However, an exhaustive analysis of this approach is out of the scopes of the present paper.

7. THE THEORY OF SEPARATION OF VARIABLES IN  $\omega\mathcal{H}$  MANIFOLDS

**7.1. Darboux-Haantjes coordinates.** The simple form admitted by the Haantjes operators (68) in AA variables suggests the search for a set of distinguished local coordinates in  $M$  that, at the same time be symplectic and diagonalize every Haantjes operator of the underlying geometric structure.

**Definition 48.** Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $\omega\mathcal{H}$  manifold. A set of local coordinates  $(q_1, \dots, q_n; p_1, \dots, p_n)$  will be said to be a set of Darboux-Haantjes (DH) coordinates if in this set the symplectic form  $\omega$  assumes the Darboux form

$$(101) \quad \omega = \sum_{i=1}^n dp_i \wedge dq_i$$

and each Haantjes operator diagonalizes:

$$(102) \quad \mathbf{K}_\alpha = \sum_{i=1}^n l_i^{(\alpha)} \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right) \quad \alpha = 0, \dots, n-1,$$

with  $l_i^{(0)} = 1, i = 1, \dots, n$ .

**Remark 49.** The set of semisimple operators is closed under  $C^\infty(M)$ -linear combinations and under the product of operators. Thus, if a set of DH coordinates exists in a given  $\omega\mathcal{H}$  manifold, the compatibility conditions (49) and (50) are automatically satisfied, thanks to Proposition 7.

There is a natural relation between AA variables and DH coordinates in the Haantjes geometry, as clarified below.

**Proposition 50.** Any set of AA variables for a completely integrable system is a set of DH coordinates for the Haantjes structure  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  given by the symplectic form  $\omega$  and the Haantjes operators (68).

In the subsequent considerations, under mild hypotheses we solve the problem of the existence of DH coordinates for a given  $\omega\mathcal{H}$  manifold, possibly different from the ones of Theorem 41.

First, we present some properties of algebraic and differential nature that hold under the assumption that the  $\omega\mathcal{H}$  structure is semisimple.

**Definition 51.** Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $\omega\mathcal{H}$  manifold. It will be said to be semisimple if each operator  $\mathbf{K}_\alpha$  is semisimple, that is each  $\mathbf{K}_\alpha$  admits a reference frame of proper eigenvectors (according to Definition 16).

**Proposition 52.** In a semisimple  $\omega\mathcal{H}$  manifold, the relations

$$(103) \quad \mathbf{\Omega}(\mathcal{D}_j^\alpha) \equiv (E_j^\alpha)^\circ \Leftrightarrow \mathcal{D}_j^\alpha \equiv \mathbf{P}((E_j^\alpha)^\circ) = (E_j^\alpha)^\perp,$$

$$(104) \quad \mathbf{\Omega}(E_j^\alpha) \equiv (\mathcal{D}_j^\alpha)^\circ \Leftrightarrow E_j^\alpha \equiv \mathbf{P}((\mathcal{D}_j^\alpha)^\circ) = (\mathcal{D}_j^\alpha)^\perp,$$

hold true.

*Proof.* Property (103) follows from the compatibility condition (48) and from the invertibility of the symplectic operator  $\mathbf{\Omega}$ . In fact, for each eigenvector  $Y_j \in \mathcal{D}_j^\alpha$ , the one-form  $\mathbf{\Omega}Y_j$  is an eigenform of  $\mathbf{K}_\alpha^T$ , as one infers from

$$\mathbf{K}_\alpha^T \mathbf{\Omega}Y_j \stackrel{(48)}{=} \mathbf{\Omega} \mathbf{K} Y_j = l_j^{(\alpha)} \mathbf{\Omega} Y_j.$$



Then, by taking into account eq. (44), we deduce that  $\Omega Y_j$  belongs to  $(E_j^\alpha)^\circ$ . Since it has the same dimension of  $\mathcal{D}_j^\alpha$ , we get eq. (103). The relation (104) follows from eq. (103) and from the observation that, by construction,  $E_j$  is a complementary subspace of  $\mathcal{D}_j^\alpha$  in  $TM$ .  $\square$

**Proposition 53.** *In a semisimple  $\omega\mathcal{H}$  manifold, the distributions  $\mathcal{D}_j^\alpha$  are integrable and even dimensional. In addition, if the  $\omega\mathcal{H}$  structure is semisimple, their integral leaves are symplectic submanifolds of  $M$  and are symplectically orthogonal to each other, namely*

$$(105) \quad \omega(\mathcal{D}_j^\alpha, \mathcal{D}_j^\alpha) = \text{symplectic}$$

$$(106) \quad \omega(\mathcal{D}_j^\alpha, \mathcal{D}_k^\alpha) = 0 \quad j \neq k$$

*Proof.* The distributions  $\mathcal{D}_j^\alpha$  are integrable due to Theorem 17 and are even-dimensional by virtue of Corollary 31. Moreover, they are symplectic as

$$\mathcal{D}_j^\alpha \cap (\mathcal{D}_j^\alpha)^\perp \stackrel{(104)}{=} \mathcal{D}_j^\alpha \cap E_j^\alpha = \{0\}.$$

Finally, property (106) follows from the fact that  $\mathcal{D}_k^\alpha \subseteq E_j^\alpha \stackrel{(104)}{=} (\mathcal{D}_j^\alpha)^\perp$ , if  $j \neq k$ .  $\square$

**7.2. Generators of a  $\omega\mathcal{H}$  manifold.** We shall investigate here the possibility that the Haantjes operators of a  $\omega\mathcal{H}$  manifold can be generated by a single Haantjes (or Nijenhuis) operator.

**Definition 54.** *Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a  $2n$ -dimensional  $\omega\mathcal{H}$  manifold. An operator  $\mathbf{K}$  will be called a generator of the  $\omega\mathcal{H}$  structure if the set*

$$(107) \quad \mathcal{B}_c := \{\mathbf{I}, \mathbf{K}, \mathbf{K}^2, \mathbf{K}^{n-1}\}$$

*is a basis of the Haantjes module  $\mathcal{K}$ . Such a basis will be called a cyclic basis of  $\mathcal{K}$  and allows one to represent each Haantjes operator  $\mathbf{K}_\alpha$  as a polynomial field in  $\mathbf{K}$  of degree at most  $(n-1)$ , i.e.*

$$(108) \quad \mathbf{K}_\alpha = p_\alpha(\mathbf{x}, \mathbf{K}) = \sum_{i=0}^{n-1} a_i^{(\alpha)}(\mathbf{x}) \mathbf{K}^i,$$

where  $a_i^{(\alpha)}(\mathbf{x})$  are smooth functions in  $M$  such that  $\det(a_i^{(\alpha)}) \neq 0$ .

**Proposition 55.** *The set of the generators of an  $\omega\mathcal{H}$  structure coincides with the set of the operators that belong to the module  $\mathcal{K}$  generated by the original Haantjes operators  $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$ , and have their minimal polynomial of degree  $n$ . In addition, if the  $\omega\mathcal{H}$  structure is semisimple, any generator of  $\mathcal{K}$  is maximal, according to Definition 32.*

*Proof.* If  $\mathbf{K}$  is a generator, together with any of its powers, it belongs to  $\mathcal{K}$  thanks to Proposition 11. Therefore, all of its powers are Haantjes operators compatible with the  $\omega\mathcal{H}$  structure. Moreover, being  $\mathcal{B}_c$  defined by (107) a basis of  $\mathcal{K}$ , the first  $(n-1)$  powers of  $\mathbf{K}$  must be linearly independent; thus its minimal polynomial has degree not lesser than  $n$ . Precisely, it has just degree  $n$ , as the powers of  $\mathbf{K}$  greater than  $(n-1)$ , belonging to  $\mathcal{K}$  as well, can be written as linear combinations of the cyclic base.

Conversely, if  $\mathbf{K}$  is an operator belonging to the module  $\mathcal{K}$ , we deduce that  $\mathbf{K}$ , together with its powers, is a Haantjes operator compatible with the  $\omega\mathcal{H}$  structure.

Since its minimal polynomial is of degree  $n$  by hypothesis, its first  $(n - 1)$  powers  $\{I, K, K^2, \dots, K^{n-1}\}$  are linearly independent; they form a basis for the module  $\mathcal{K}$  and consequently generate  $\mathcal{K}$ .

Finally, if the  $\omega\mathcal{H}$  structure is semisimple, the Haantjes operators  $K_\alpha$  admit a common reference frame (thanks to their commutativity (49)) where they take a diagonal form, and each element of the module  $\mathcal{K}$  is semisimple as well in the same reference frame. Therefore, any generator is semisimple and, having the minimal polynomial of degree  $n$ , is maximal by virtue of Lemma 33.  $\square$

Let us denote with  $D_j$  the eigen-distributions of a generator  $K$ , and recall that  $D_j^\alpha$  denote the eigen-distributions of the Haantjes operators  $K_\alpha$ .

**Proposition 56.** *Let  $(M, \omega, K_0, K_1, \dots, K_{n-1})$  be a  $2n$ -dimensional, semisimple  $\omega\mathcal{H}$  manifold. If the Haantjes module  $\mathcal{K}$  admits a generator  $K$ , then the following alternative conditions hold*

$$(109) \quad D_j \cap D_k^\alpha = \{0\} \quad \text{or} \quad D_j \subseteq D_k^\alpha, \quad \alpha = 0, \dots, n-1, \quad k = 1, \dots, n.$$

*Proof.* Let us consider the eigen-distributions  $D_j := \text{Ker}(K - l_j(\mathbf{x})I)$ , for  $j = 1, \dots, n$ , which are two-dimensional by the maximality property of any generator. As  $K$  and  $K_\alpha$  commute by the hypothesis (49),  $D_j$  is  $K_\alpha$ -invariant, therefore  $K_\alpha$  can be restricted to  $D_j$  and such a restriction is semisimple. Consequently, in  $D_j$  there exist two independent eigenvector fields  $Y_1, Y_2$  of  $K_\alpha$ . Let us show that they correspond to the same eigenvalue of  $K_\alpha$ , say  $l_k^{(\alpha)}$ . Indeed, if by absurd should they correspond to different eigenvalues of  $K_\alpha$ , the property (106) would imply that

$$\omega(Y_1, Y_2) = 0.$$

Nevertheless, being  $D_j$  two-dimensional, this should imply that  $D_j$  is isotropic, in contradiction with the property to be symplectic (105). Hence the thesis.  $\square$

**7.3. Existence of DH coordinates.** We shall prove that, in a given semisimple  $\omega\mathcal{H}$  manifold, the existence of a Haantjes (or Nijenhuis) generator is equivalent to the existence of a set of DH coordinates.

The main result of this section is the following theorem, establishing the existence of DH coordinates.

**Theorem 57.** *Let  $(M, \omega, K_0, K_1, \dots, K_{n-1})$  be a  $2n$ -dimensional, semisimple  $\omega\mathcal{H}$  manifold. If a set of DH coordinates and  $\{(\mathbf{q}, \mathbf{p})\}$  does exist, then each operator of the form*

$$(110) \quad K = \sum_{i=1}^n \lambda_i(\mathbf{x}) \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right),$$

( $\lambda_i(\mathbf{x})$  being arbitrary smooth functions on  $M$  fulfilling the maximality condition  $\lambda_i(\mathbf{x}) \neq \lambda_j(\mathbf{x}), \forall \mathbf{x} \in M$ ) is a generator of the given Haantjes structure through the relations

$$(111) \quad K_\alpha = \sum_{i=1}^n l_i^{(\alpha)} \frac{\Pi_{j \neq i}(K - \lambda_j I)}{\Pi_{j \neq i}(\lambda_i - \lambda_j)} \quad \alpha = 0, \dots, n-1.$$

Conversely, if the module  $\mathcal{K}$  generated by  $\{K_0, K_1, \dots, K_{n-1}\}$  contains a maximal operator, then there exists locally a set of Darboux-Haantjes coordinates.

*Proof.* In each set of DH coordinates, the Haantjes operators  $\{\mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$  assume a diagonal form. Let us take as possible generators the family of diagonal operators (110). They are Haantjes operators compatible with  $\omega$  and with  $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$  thanks to the Remark 49. In order to assure that  $\mathbf{K}_\alpha$  is generated by  $\mathbf{K}$ , according to eq. (108), it suffices to show that there exist  $(n-1)$  polynomial fields  $p_\alpha(\mathbf{x}, \lambda)$ ,  $\lambda \in \mathbb{R}$ , such that

$$l_i^{(\alpha)}(\mathbf{x}) = p_\alpha(\mathbf{x}, \lambda_i) = \sum_{k=0}^{n-1} c_k^{(\alpha)}(\mathbf{x}) \lambda_i^k(\mathbf{x}), \quad \alpha = 1, \dots, n-1.$$

These algebraic equations are solved by means of the Lagrange interpolation polynomials of degree  $(n-1)$

$$\pi_i(\lambda) = \frac{\prod_{j \neq i} (\lambda - \lambda_j)}{\prod_{j \neq i} (\lambda_i - \lambda_j)},$$

which yield the expressions

$$p_\alpha(\mathbf{x}, \lambda) = \sum_{i=1}^n l_i^{(\alpha)}(\mathbf{x}) \pi_i(\lambda).$$

Therefore,

$$(112) \quad \mathbf{K}_\alpha = \sum_{i=1}^n p_\alpha(\mathbf{x}, \lambda_i) \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right) = p_\alpha(\mathbf{x}, \mathbf{K}) \quad \alpha = 0, \dots, n-1.$$

Conversely, if the module  $\mathcal{K}$  contains a maximal operator  $\mathbf{K}$ , such an operator is a generator of the  $\omega\mathcal{H}$  structure thanks to Proposition 55. Moreover, being  $\mathbf{K}$  a maximal Haantjes operator, we show that  $\mathbf{K}$  admits a set of symplectic coordinates, in which it takes the diagonal form (110). Indeed, we need to prove that a parametrization of the characteristic web of  $\mathbf{K}$  exists with  $2n$  coordinate functions  $(q_1, \dots, q_n; p_1, \dots, p_n)$  that are Darboux coordinates for  $\omega$ . Under the maximality assumption, the characteristic web is of type  $(n, n, 2)$ , with characteristic fibers of co-dimension 2 (see [2]). Furthermore, the distributions  $E_j$ ,  $(E_j)^\circ$  are integrable; we shall see that their foliations  $(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s)$  form the characteristic web of the  $\omega\mathcal{H}$  manifold. Equivalently, any parametrization of the web always diagonalizes simultaneously each operator  $\mathbf{K}_\alpha$  and  $\mathbf{K}_\alpha^T$ .

Each characteristic distribution  $E_k$  is integrable and is of co-dimension 2. One can find in each annihilator  $(E_k)^\circ$  the differentials of two functions  $(f_k, g_k)$  that span  $(E_k)^\circ$ . Collecting together these functions, one obtains a parametrization  $(f_1, \dots, f_n; g_1, \dots, g_n)$  of the characteristic web of  $\mathbf{K}$ . By virtue of Corollary 24, such parametrization enables us to put the operator  $\mathbf{K}$  in diagonal form, whilst the symplectic form reads

$$(113) \quad \omega = \sum_{i=1}^n b_i(f_i, g_i) dg_i \wedge df_i.$$

In fact, as  $\frac{\partial}{\partial f_j}, \frac{\partial}{\partial g_j} \in \mathcal{D}_j$  due to relation (42), eq. (106) implies that

$$0 = \omega\left(\frac{\partial}{\partial f_j}, \frac{\partial}{\partial f_k}\right) = \omega\left(\frac{\partial}{\partial g_j}, \frac{\partial}{\partial g_k}\right) = \omega\left(\frac{\partial}{\partial f_j}, \frac{\partial}{\partial g_k}\right), \quad j \neq k.$$

Furthermore, the closure of  $\omega$  implies that the component functions  $b_i$  depend only on the pair  $(f_i, g_i)$ . Then, by means of eq. (105), inside the subspace  $(E_j)^\circ$ ,

where the restriction of  $\mathbf{K}^T$  is a multiple of the identity thanks to the condition (109), one can perform a Darboux transformation  $q_i = \tilde{f}_i(f_i, g_i), p_i = \tilde{g}_i(f_i, g_i)$ , involving only the pairs  $(f_j, g_j)$ . After these transformations, one obtains a local chart  $\{(q_1, \dots, q_n; p_1, \dots, p_n)\}$  in which the symplectic form  $\omega$  takes the Darboux form (101). Therefore, we have proven the existence of a coordinate system  $(\mathbf{q}; \mathbf{p})$  which still diagonalizes the generator  $\mathbf{K}$  and simultaneously reduces  $\omega$  to a canonical form.

Finally, even the original Haantjes operators  $\mathbf{K}_\alpha$ , being generated by  $\mathbf{K}$  according to eq. (108), in such coordinates take the diagonal form (112). We conclude that these coordinates are DH coordinates in the considered  $\omega\mathcal{H}$  manifold, for each Haantjes operator belonging to the module  $\mathcal{K}$ .  $\square$

As an immediate consequence of the previous Theorem, of Proposition 26 and of Proposition 9, we have the following

**Corollary 58.** *In any set of DH coordinates, a function  $f_i$  is a characteristic function of the Haantjes web, related to the eigenvalues  $\{l_i^{(\alpha)}\}_{1 \leq \alpha \leq (n-1)}$ , if and only if it depends on the single pair  $(q_i, p_i)$  only, that is*

$$(114) \quad f_i = f_i(q_i, p_i) .$$

Therefore,  $(f_i, f_j)$  satisfy the involution relations

$$(115) \quad \{f_i, f_j\} = 0 \quad i \neq j .$$

Moreover, a family of generators of the Haantjes module does exist, formed by Nijenhuis operators. They are the operators that in DH coordinates take the form (110), with eigenvalues

$$(116) \quad \lambda_i(\mathbf{x}) = \lambda_i(q_i, p_i) \quad i = 1, \dots, n ,$$

where  $\lambda_i$  is an arbitrary smooth function depending only on the single conjugate pair  $(q_i, p_i)$ . Therefore, such eigenvalues are characteristic functions of the Haantjes web and are in mutually involution.

The Haantjes generators (110) are parametrized by the arbitrary functions  $\lambda_i(\mathbf{x})$ ; therefore, they form an infinite family of operators. By contrast, given a set of  $n$  functions in involution  $(H_1, \dots, H_n)$ , the Haantjes operators providing the associated Lenard-Haantjes chain (53) are uniquely determined, once a holonomic frame of common eigenvectors has been chosen. Thus, the Haantjes operators  $\{\tilde{\mathbf{K}}_0, \tilde{\mathbf{K}}_1, \dots, \tilde{\mathbf{K}}_{n-1}\}$  appear to play a more fundamental role than their generators  $\mathbf{K}$ .

**7.4. Haantjes theorem for separable systems.** Without loss of generality, we assume, that we have already exchanged the original operator  $\mathbf{K}_\alpha$  with  $\tilde{\mathbf{K}}_\alpha$  and, for the sake of simplicity, we drop off the *tilde* over  $\tilde{\mathbf{K}}_\alpha$  from now on.

The next theorem is the main result concerning the existence of separation variables for  $\omega\mathcal{H}$  manifolds. It states that such structures characterize each separable Hamiltonian system.

**Theorem 59.** *Let  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  be a semisimple  $\omega\mathcal{H}$  manifold and  $(H_1, H_2, \dots, H_n)$  be a set of  $n$  functions belonging to a Lenard-Haantjes chain generated by  $H = H_1$ . If the Haantjes module  $\mathcal{K}$  contains a maximal operator, then each set  $(\mathbf{q}, \mathbf{p})$  of DH coordinates provides separation variables for the Hamilton-Jacobi equations associated with each function  $H_j$ .*

Conversely, if  $(H_1, H_2, \dots, H_n)$  are a set of  $n$  vertically independent functions, that is

$$(117) \quad \det \left( \frac{\partial H_i}{\partial p_j} \right) \neq 0 ,$$

and are separable in a set of Darboux coordinates  $(\mathbf{q}, \mathbf{p})$ , then they belong to a Lenard–Haantjes chain w.r.t. the  $\omega\mathcal{H}$  structure  $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$  given by

$$(118) \quad \mathbf{K}_\alpha = \sum_{i=1}^n \frac{\frac{\partial H_{\alpha+1}}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right) \quad \alpha = 0, \dots, n-1 .$$

*Proof.* Theorem 57 guarantees the existence of sets of DH coordinates. Therefore it suffices to show that the functions  $H_j$  are in separable involution in these coordinates, according to eq. (1). To this aim, let us note that, due to the diagonal form (102) of  $\mathbf{K}_\alpha^T$  in a DH local chart, the relations

$$(119) \quad \frac{\partial H_j}{\partial q_k} = l_k^{(j-1)} \frac{\partial H}{\partial q_k},$$

$$(120) \quad \frac{\partial H_j}{\partial p_k} = l_k^{(j-1)} \frac{\partial H}{\partial p_k},$$

hold. Here  $l_k^{(j-1)}$  denotes the eigenvalues of the Haantjes tensors  $\mathbf{K}_\alpha^T$ ,  $\alpha = j-1$ . Therefore,

$$\{H_i, H_j\}_{|k} = l_k^{(i-1)} \frac{\partial H}{\partial q_k} l_k^{(j-1)} \frac{\partial H}{\partial p_k} - l_k^{(j-1)} \frac{\partial H}{\partial q_k} l_k^{(i-1)} \frac{\partial H}{\partial p_k} = 0 .$$

We prove the converse statement in a way analogous to that of Theorem 41. Without loss of generality, we can assume that  $\frac{\partial H}{\partial p_i} \neq 0$ ,  $i = 1, \dots, n$ . The operators (118), being diagonal in the separated coordinates, have their Haantjes tensor vanishing. Also, they commute with each others and fulfill the differential compatibility condition (50). The algebraic compatibility relations (48) with the symplectic form are equivalent to the conditions

$$(121) \quad l_{n+i}^{(\alpha)} = l_i^{(\alpha)} \quad i = 1, \dots, n .$$

Thus, the Haantjes operators (118) must possess eigenvalues that are at least double.

Finally, we impose that the integrals of motion  $(H_1, H_2, \dots, H_n)$  form a Lenard Haantjes chain generated by  $H \equiv H_1$ . Being  $\mathbf{K}_\alpha$  diagonal in the  $(\mathbf{q}, \mathbf{p})$  variables, such conditions are equivalent, for each  $\alpha$ , to the overdetermined system of  $2n$  algebraic equations in the  $n$  indeterminate functions  $l_i^{(\alpha)}$

$$(122) \quad l_i^{(\alpha)} \frac{\partial H}{\partial q_i} = \frac{\partial H_{\alpha+1}}{\partial q_i} ,$$

$$(123) \quad l_i^{(\alpha)} \frac{\partial H}{\partial p_i} = \frac{\partial H_{\alpha+1}}{\partial p_i} ,$$

$i = 1, \dots, n$ . However, the above equations are compatible, because the Benenti conditions (1) of separate involution assure that

$$\frac{\partial H}{\partial q_i} \frac{\partial H_{\alpha+1}}{\partial p_i} = \frac{\partial H}{\partial p_i} \frac{\partial H_{\alpha+1}}{\partial q_i} , \quad 1 \leq i \leq n .$$

Consequently, the equations (123) provide the unique solution (118), where the Haantjes operators  $\mathbf{K}_\alpha$  are independent thanks to the condition (117).  $\square$

Constructing explicitly a set of DH coordinates is a difficult task that entails to integrate the eigen-distributions of a Haantjes generator  $\mathbf{K}$  for the Haantjes module generated by  $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$ . We will use this method in Subsect. 9.2. However, it can be simplified when a *Nijenhuis* generator  $\mathbf{N}$  is at our disposal. In fact, the non constant eigenvalues of such  $\mathbf{N}$  are just characteristic functions of the Haantjes web by virtue of Corol. 58. Therefore, if all the eigenvalues are not constant and functionally independent, being  $\mathbf{N}$  maximal, one can start with half of the DH coordinates  $(\lambda_1(\mathbf{x}), \dots, \lambda_n(\mathbf{x}))$ . Moreover, it has been proved in [42] that they can be complemented by quadratures with  $n$  conjugated momenta  $(\mu_1(\mathbf{x}), \dots, \mu_n(\mathbf{x}))$  satisfying

$$(124) \quad \mathbf{N}^T d\mu_i = \lambda_i d\mu_i.$$

Therefore, in such canonical coordinates  $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ , the Nijenhuis generator  $\mathbf{N}$  takes a diagonal form, consequently they are DH coordinates for the given  $\omega H$  structure. They have been called Darboux-Nijenhuis (DN) coordinates in [17] and special DN coordinates in [19]. Let us note that, in such a situation, the tensor field  $\mathbf{P}_1 := \mathbf{N}\Omega^{-1}$  turns out to be a Poisson bivector compatible with  $\Omega$ , endowing the  $\omega H$  manifold  $M$  also with a bi-Hamiltonian structure.

**7.5. Classical separable Hamiltonian systems.** In this Section, given a generic Hamiltonian system whose Hamilton-Jacobi equation can be solved by separation of variables, we construct an associated Haantjes structure.

Hamiltonian systems solvable by separation of variables are obviously integrable by quadrature. Theorem 41 holds for non degenerate Hamiltonian systems which, consequently, admit the Haantjes structure (68). However, we will prove that a Haantjes structure can also be constructed starting with separation variables different from the AA variables. Moreover, such a structure does not necessarily coincide with the one coming from (68).

As an application of the defining relations (118), we get explicitly the Haantjes structure for three large classes of separable Hamiltonian systems (see [3], Proposition 5, page 125).

**Remark 60.** *For the families of  $n$ -dimensional Hamiltonian systems treated below every Haantjes operator for  $n > 2$  is not maximal. However, the associated Haantjes module  $\mathcal{K}$  contains, by construction, at least a maximal generator of the form (110).*

The proof of the statements presented hereafter is direct and is left to the reader.

**Proposition 61** (*“Functionally separated” systems*). *Let us consider the Hamiltonian function*

$$(125) \quad H(f_1(q_1, p_1), f_2(q_2, p_2), \dots, f_n(q_n, p_n)),$$

*together with the integrals of motion*

$$(126) \quad I_j = f_j(q_j, p_j), \quad j = 1, \dots, n-1.$$

*This system admits the Haantjes structure given by the Haantjes operators*

$$(127) \quad \mathbf{K}_0 = \mathbf{I}, \quad \mathbf{K}_j := \left( \frac{\partial H}{\partial f_j} \right)^{-1} \left( \frac{\partial}{\partial q_j} \otimes dq_j + \frac{\partial}{\partial p_j} \otimes dp_j \right), \quad j = 1, \dots, n-1.$$

The spectrum of  $\mathbf{K}_j$  is

$$(128) \quad \text{Spec}(\mathbf{K}_j) = \left\{ 0, \left( \frac{\partial H}{\partial f_j} \right)^{-1} \right\} .$$

Such Haantjes operators provide the Lenard–Haantjes chain

$$\mathbf{K}_j^T dH = dI_j \quad j = 1, \dots, n-1, \quad I_0 = H .$$

**Example 62.** The expression of the natural Hamiltonian function additively separated in cartesian coordinates reads

$$H = \sum_{j=1}^n \left( \frac{p_j^2}{2m_j} + V(x_j) \right) .$$

It admits the integrals of motion

$$I_j = \frac{p_j^2}{2m_j} + V(x_j) \quad j = 1, \dots, n-1 .$$

The Haantjes operators are  $\mathbf{K}_0 = \mathbf{I}$ , and the uniform ones

$$\mathbf{K}_j := \left( \frac{\partial}{\partial q_j} \otimes dq_j + \frac{\partial}{\partial p_j} \otimes dp_j \right) \quad j = 1, \dots, n-1 .$$

**Proposition 63** (Telescopic systems). The system defined by the Hamiltonian function

$$(129) \quad H \left( f_n \left( f_{n-1} \left( \dots f_2(f_1(q_1, p_1), q_2, p_2), \dots, q_{n-1}, p_{n-1} \right), q_n, p_n \right) \right)$$

together with the integrals of motion

$$(130) \quad I_1 = f_1(q_1, p_1), \quad I_2 = f_2(f_1(q_1, p_1), q_2, p_2), \quad I_n := H ,$$

admits the Haantjes structure given by the Haantjes operators  $\mathbf{K}_0 = \mathbf{I}$  and

$$(131) \quad \mathbf{K}_j := \left( \frac{\partial f_n}{\partial f_{n-1}} \frac{\partial f_{n-1}}{\partial f_{n-2}} \dots \frac{\partial f_{j+1}}{\partial f_j} \right)^{-1} \sum_{i \leq j} \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right) \quad j = 1, \dots, n-1 .$$

The spectrum of these operators is given by

$$(132) \quad \text{Spec}(\mathbf{K}_j) = \left\{ 0, \left( \frac{\partial f_n}{\partial f_{n-1}} \frac{\partial f_{n-1}}{\partial f_{n-2}} \dots \frac{\partial f_{j+1}}{\partial f_j} \right)^{-1} \right\} .$$

The operators  $\mathbf{K}_j$  form the Lenard–Haantjes chain

$$\mathbf{K}_j^T dH = dI_j \quad j = 1, \dots, n-1 .$$

**Example 64.** The most general Hamiltonian function separable in spherical coordinates  $\{(q_1 = \phi, q_2 = \theta, q_3 = r)\}$  is given by

$$(133) \quad H = \frac{1}{2m} \left( p_r + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + g(r) + \frac{h(\theta)}{r^2} + \frac{s(\phi)}{r^2 \sin^2 \theta},$$

where  $g(r), h(\theta), s(\phi)$  are arbitrary smooth functions of their argument. The integrals of motion are

$$I_1 = \frac{p_\phi^2}{2m} + s(\phi), \quad I_2 = \frac{1}{2m} \left( p_\theta^2 + \frac{p_\phi^2 + s(\phi)}{\sin^2 \theta} \right) + h(\theta), \quad I_3 = H .$$

According to eqs. (131), the Haantjes operators are  $\mathbf{K}_0 = \mathbf{I}$ , and

$$(134) \quad \mathbf{K}_1 = r^2 \sin^2 \theta \left( \frac{\partial}{\partial \phi} \otimes d\phi + \frac{\partial}{\partial p_\phi} \otimes dp_\phi \right)$$

$$(135) \quad \mathbf{K}_2 = r^2 \left( \frac{\partial}{\partial \phi} \otimes d\phi + \frac{\partial}{\partial p_\phi} \otimes dp_\phi + \frac{\partial}{\partial \theta} \otimes d\theta + \frac{\partial}{\partial p_\theta} \otimes dp_\theta \right).$$

**Remark 65.** If  $g = -k/r$ ,  $h(\theta) = s(\phi) = 0$ , one gets the classical Coulomb-Kepler system. For such model, the authors of [44] proved that it does not exist a Nijenhuis recursion operator compatible with the original symplectic structure. Indeed, the Nijenhuis structure they found provides a Lenard chain of dependent integrals of motion.

Accordingly, we stress the fact that a Haantjes structure is needed to construct the Lenard–Haantjes chain formed by

$$H_1 = H, \quad H_2 = I_1, \quad H_3 = I_2.$$

**Proposition 66** (Gantmacher systems). Let us consider the Hamiltonian system defined by the Hamiltonian function

$$(136) \quad H = \frac{\sum_{k=1}^n f_k(q_k, p_k)}{\sum_{k=1}^n g_k(q_k, p_k)},$$

with the integrals of motion

$$(137) \quad I_k = f_k(q_k, p_k) - H g_k(q_k, p_k), \quad k = 1, \dots, n-1.$$

The system admits the Haantjes structure given by the Haantjes operators  $\mathbf{K}_0 = \mathbf{I}$ , and

$$(138) \quad \mathbf{K}_j := -g_j \mathbf{I} + G \left( \frac{\partial}{\partial q_j} \otimes dq_j + \frac{\partial}{\partial p_j} \otimes dp_j \right) \quad j = 1, \dots, n-1,$$

with  $G := \sum_{k=1}^n g_k(q_k, p_k)$ , whose spectrum is given by

$$(139) \quad \text{Spec}(\mathbf{K}_j) = \{-g_j, G - g_j\}.$$

They provide the Lenard–Haantjes chain

$$\mathbf{K}_j^T dH = dI_j \quad j = 1, \dots, n-1.$$

**Example 67** (Liouville systems). Choosing in eq. (136)

$$f_k := \frac{1}{2} a_k(q_k) p_k^2 + V_k(q_k), \quad g_k := g_k(q_k),$$

one gets the class of systems that Liouville showed to be separable [54]. For instance, they include each natural Hamiltonian system in the plane, separable in cartesian  $(x, y; p_x, p_y)$ , polar  $(r, \theta; p_r, p_\theta)$ , elliptic or parabolic coordinates  $(\xi, \eta; p_\xi, p_\eta)$ . The associated Haantjes operators read, respectively



$$(140) \mathbf{K}_{car} = -\left(\frac{\partial}{\partial y} \otimes dy + \frac{\partial}{\partial p_y} \otimes dp_y\right),$$

$$(141) \mathbf{K}_{pol} = -r^2 \left(\frac{\partial}{\partial \theta} \otimes d\theta + \frac{\partial}{\partial p_\theta} \otimes dp_\theta\right),$$

$$(142) \mathbf{K}_{par} = \eta^2 \left(\frac{\partial}{\partial \xi} \otimes d\xi + \frac{\partial}{\partial p_\xi} \otimes dp_\xi\right) - \xi^2 \left(\frac{\partial}{\partial \eta} \otimes d\eta + \frac{\partial}{\partial p_\eta} \otimes dp_\eta\right),$$

$$(143) \mathbf{K}_{ell} = -\eta^2 \left(\frac{\partial}{\partial \xi} \otimes d\xi + \frac{\partial}{\partial p_\xi} \otimes dp_\xi\right) - \xi^2 \left(\frac{\partial}{\partial \eta} \otimes d\eta + \frac{\partial}{\partial p_\eta} \otimes dp_\eta\right).$$

**Remark 68.** *The Haantjes operators (141), (142), (143), obtained through the construction (118) do not coincide with the Haantjes operators of the same system that one could construct through AA variables according to eq. (68). In fact, as a consequence of Remark 43, the eigenvalues of (118) must be integrals of motion, whilst the eigenvalues of the operators (141), (142), (143) are not invariant along the flow of the Hamiltonian vector fields. This fact implies that Haantjes structures for integrable systems are usually not unique.*

## 8. THE INVERSE PROBLEM FOR SYSTEMS WITH TWO DEGREES OF FREEDOM

In this Section, we are concerned with the inverse problem. In other words, given a set of independent functions in involution, we will construct a Haantjes structure for them, that is we shall determine the Haantjes structures of an assigned integrable system.

**8.1. A general procedure.** Let us consider the simplest case of Hamiltonian systems with two degrees of freedom. We propose a general procedure to compute a Haantjes operator adapted to the Lenard chain formed by two integrals of motion. We search for a generator of the Haantjes module, that is, a Haantjes operator  $\mathbf{K}$  whose minimal polynomial should be of degree two, namely, the maximum degree allowed by our assumptions:

$$(144) \quad m_{\mathbf{K}}(\lambda) := \lambda^2 - c_1(\mathbf{x})\lambda - c_2(\mathbf{x}).$$

Let us note that such a request does not imply the semisimplicity property for  $\mathbf{K}$ , unless the existence of two *distinct* roots of  $m_{\mathbf{K}}(z)$  is also supposed.

**Remark 69.** *In the case  $n = 2$ , every pair of Haantjes operators  $\{\mathbf{I}, \mathbf{K}\}$  generates a Haantjes module (see Proposition 10) and a ring w.r.t. the operator product (Proposition 11). Any other generator  $\tilde{\mathbf{K}}$  has the form*

$$(145) \quad \tilde{\mathbf{K}} = f\mathbf{I} + g\mathbf{K},$$

where  $f$  is an arbitrary smooth function and  $g$  a nowhere vanishing smooth function. Thus,

$$(146) \quad \det(\tilde{\mathbf{K}} - \lambda\mathbf{I}) = \det(f\mathbf{I} + g\mathbf{K} - \lambda\mathbf{I}) = g^n \det\left(\mathbf{K} - \frac{\lambda - f}{g}\mathbf{I}\right).$$

Therefore, the eigenvalues  $\lambda_i$  of  $\tilde{\mathbf{K}}$  and  $l_i$  of  $\mathbf{K}$  are related by the affine equation

$$(147) \quad \lambda_i = f + g l_i \quad i = 1, 2;$$

consequently,  $\lambda_1 \equiv \lambda_2$  is equivalent to  $l_1 \equiv l_2$ . Then, we can conclude that the Haantjes module contains a maximal generator if and only if all generators are maximal.

The procedure entails two steps.

- (1) Given two independent integrals of motion in involution  $(H_1, H_2)$ , find a Haantjes operator that provide a Lenard-Haantjes chain for them, therefore fulfilling the conditions (61). Consequently, find an operator  $\mathbf{K}$  such that

$$(148) \quad \mathbf{K}^T \Omega = \Omega \mathbf{K}$$

$$(149) \quad \mathbf{K}^T dH_1 = dH_2$$

$$(150) \quad (\mathbf{K}^T)^2 dH_1 = (c_1 \mathbf{K}^T + c_2 \mathbf{I}) dH_1$$

$$(151) \quad \mathcal{H}_{\mathbf{K}}(X, Y) = 0 \quad \forall X, Y \in TM$$

The algebraic compatibility condition (148) allows to reduce the unknown components of the  $\mathbf{K}$  operator from 16 to 6. Therefore

$$(152) \quad \mathbf{K} = \left[ \begin{array}{cc|cc} l_1^1 & l_2^1 & 0 & l_4^1 \\ l_1^2 & l_2^2 & -l_4^1 & 0 \\ \hline 0 & l_2^3 & l_1^1 & l_1^2 \\ -l_2^3 & 0 & l_2^1 & l_2^2 \end{array} \right],$$

where  $l_j^i$  are arbitrary functions on  $M$ . The conditions (149) and (150) provide a system of 8 algebraic equations in the 6 unknown functions, being  $c_1 = l_1^1 + l_2^2$  and  $c_2 = -l_1^1 l_2^2 + l_2^1 l_1^2 - l_4^1 l_2^3$ . As such equations are not independent, we are left with 3 unknown functions. The vanishing of the Haantjes tensor of  $\mathbf{K}$  (151) provides an over-determined system of 24 PDEs of first order, which can be managed with some suitable ansatz. For instance, some homogeneity properties for the components of  $\mathbf{K}$  can be assumed.

The next step allows to construct DH coordinates.

- (2) If the Haantjes generator  $\mathbf{K}$  found is maximal, it does exist also a Nijenhuis generator  $\mathbf{N}$ . Then, find two smooth functions  $(f, g)$ , with  $g$  nowhere vanishing, such that

$$(153) \quad \mathbf{N} = f \mathbf{I} + g \mathbf{K}$$

$$(154) \quad \mathcal{T}_{\mathbf{N}}(X, Y) = 0 \quad \forall X, Y \in TM$$

The vanishing of the Nijenhuis torsion furnishes an over-determined system of 24 equations with the two unknown functions  $(f, g)$ .

If the above procedure is successful, the eigenvalues (147) of  $\mathbf{N}$  are half of the DH coordinates that we are looking for, by virtue of Rem. 27. A set of conjugate momenta can be found as characteristic functions of the Haantjes web as we shall do in Subsect. 9.2.2.

**Remark 70.** For a large class of systems, i.e. the so called quasi-bi-Hamiltonian systems [14, 48, 49, 62], step 2 can be simplified. In fact, thanks to the results of [63], we can prove that if eqs. (153), (154) admit the solution

$$(155) \quad f = \frac{1}{2} \text{trace}(\mathbf{K}), \quad g = -1,$$

then the Hamiltonian system under scrutiny has a quasi-bi-Hamiltonian formulation. Therefore, the eigenvalues of the Haantjes generator  $\mathbf{K}$ , found in step 1, are themselves characteristic functions of the web since, by plugging the solution (155) into eq. (147), we obtain that

$$(156) \quad l_1 = \lambda_2, \quad l_2 = \lambda_1.$$

This property will be useful in Section 8.

Let us note that eq. (19b), for  $n = 2$ , coincides with the projection of eq. (153), with the solution (155), from the tangent bundle  $T^*\mathcal{E}_2$  onto  $\mathcal{E}_2$ .

**8.2. On the super-integrable Post-Winternitz system.** In this section, by means of the procedure described above we face the inverse problem for a system which recently has attracted much attention: the Post-Winternitz (PW) system [55]. Indeed, it is a maximally *superintegrable* system [47] with integrals of motion cubic and quartic in the momenta. As a consequence, its bounded orbits are closed and periodic. Thus, as well as every superintegrable system, it does not fulfill the non degeneracy condition (67) and Theorem 41 cannot be applied. Despite its regularity properties, the separability structures of the PW system are not known. Since it does not belong to the Stäckel class, the PW system is certainly not separable by an extended point transformation.

Let us consider the Hamiltonian system

$$(157) \quad H = \frac{1}{2}(p_x^2 + p_y^2) + a \frac{x}{y^{\frac{2}{3}}} \quad a \in \mathbb{R},$$

with the two independent integrals of motion

$$(158) \quad H_2 = 2p_x^3 + 3p_y^2 p_x + a \left( 9y^{\frac{1}{3}} p_y + 6 \frac{x}{y^{\frac{2}{3}}} p_x \right),$$

$$(159) \quad H_3 = p_y^4 - 12ay^{\frac{1}{3}} p_x p_y + 4a \frac{x}{y^{\frac{2}{3}}} p_y^2 - 2a^2 \left( 9y^{2/3} - \frac{2x^2}{y^{\frac{4}{3}}} \right).$$

We shall prove that they form two different Lenard-Haantjes chains  $(H, H_2)$  and  $(H, H_3)$ , each of them being sufficient to assure the complete integrability of the PW system.

By performing the extended-point canonical transformation

$$(160) \quad q_1 = y^{\frac{1}{3}}, \quad q_2 = \frac{x}{y^{\frac{2}{3}}}, \quad p_1 = 2 \frac{x}{y^{\frac{1}{3}}} p_x + 3y^{\frac{2}{3}} p_y, \quad p_2 = y^{\frac{2}{3}} p_x,$$

we reduce the Hamiltonian functions to a rational form from which we infer the weights of the three components of (152), still unknown after having imposed the conditions (149), (150). As a result of the previous approach, we get the Haantjes structure  $(\omega, \mathbf{I}, \mathbf{K}_{PW}^{(2)})$  for the Lenard-Haantjes chain  $(H, H_2)$ , where

$$(161) \quad \mathbf{K}_{PW}^{(2)} = 3 \left[ \begin{array}{cc|cc} 2p_x & p_y & 0 & 3y \\ 0 & 2p_x & -3y & 0 \\ \hline 0 & 0 & 2p_x & 0 \\ 0 & 0 & p_y & 2p_x \end{array} \right].$$

Similarly, we obtain the Haantjes structure  $(\omega, \mathbf{I}, \mathbf{K}_{PW}^{(3)})$  for the Lenard-Haantjes chain  $(H, H_3)$ , where

(162)

$$\mathbf{K}_{PW}^{(3)} = 4 \left[ \begin{array}{cc|cc} p_y^2 + 2a\frac{x}{y^{2/3}} & -(p_x p_y + 3ay^{1/3}) & 0 & -3yp_x \\ 0 & p_y^2 + 2a\frac{x}{y^{2/3}} & 3yp_x & 0 \\ \hline 0 & 0 & p_y^2 + 2a\frac{x}{y^{2/3}} & 0 \\ 0 & 0 & -(p_x p_y + 3ay^{1/3}) & p_y^2 + 2a\frac{x}{y^{2/3}} \end{array} \right].$$

Although both  $\mathbf{K}_{PW}^{(2)}$  and  $\mathbf{K}_{PW}^{(3)}$  have their minimal polynomial of degree 2, they are not semisimple, since each of them has only one eigenvalue of algebraic multiplicity equal to 4, with two proper eigenvectors and two generalized eigenvectors. Therefore, they do not fulfill the assumptions of Theor. 57. Moreover, neither of the two Haantjes modules generated by  $\{\mathbf{I}, \mathbf{K}_{PW}^{(1)}\}$  and  $\{\mathbf{I}, \mathbf{K}_{PW}^{(2)}\}$  possesses a maximal generator by Remark 69. However, their two Lenard-Haantjes chain ensure the superintegrability of the PW model. Thus, the existence of  $\mathbf{K}_{PW}^{(2)}$  and  $\mathbf{K}_{PW}^{(3)}$  shows that the Haantjes theory can be naturally applied to non-Stäckel systems not possessing any evident separability structure, even when they do not satisfy the nondegeneracy condition (67).

## 9. APPLICATIONS TO SEPARABLE HAMILTONIAN SYSTEMS

In this section, in order to show the large range of applicability of the theory previously developed, we will discuss two important examples of integrable systems. The first one concerns a Hamiltonian system on a six-dimensional symplectic manifold, which is obtained as a stationary reduction of the seventh order equation of the Korteweg de Vries (KdV) hierarchy. The second one is a Drach-Holt type system, considered to be an example of *nonseparable system* till now.

### 9.1. The stationary reduction of the seventh order KdV flow revisited.

In [60], a method to obtain the Poisson pencil  $P_1 - \lambda P_0$  of the stationary flows of the KdV hierarchy was presented. In [48], this method was applied to get the stationary reduction of the seventh order equation of the hierarchy. The restricted Poisson pencil turns out to be a degenerate pencil of co-rank one in a seven dimensional manifold  $\mathcal{M}^{(7)}$ , being therefore a Gelfand-Zakarevich system [24]. It possesses a polynomial Casimir function of length four, starting with a Casimir of  $P_0$  and ending with a Casimir of  $P_1$ . Then, a Marsden-Ratiu reduction procedure [45], similar to the one used in other cases [60, 50, 51, 18, 20], was performed to each six-dimensional symplectic leaf  $S_0$  of the Poisson tensor  $P_0$ , in order to get rid of the Casimir of  $P_0$ . Furthermore, by restricting the polynomial Casimir function to  $S_0$ , one of the authors was able to obtain in [48] three Hamiltonian functions in involution in the Darboux chart  $(q_1, q_2, q_3, p_1, p_2, p_3)$

(163)

$$\begin{aligned} H_1 &= p_1 p_2 + \frac{1}{2} p_3^2 - \frac{5}{8} q_1^4 + \frac{5}{2} q_1^2 q_2 + \frac{1}{2} q_1 q_3^2 - \frac{1}{2} q_2^2, \\ H_2 &= \frac{1}{2} p_1^2 + p_1 p_2 q_1 + p_3^2 q_1 - p_2^2 q_2 - p_2 p_3 q_3 - \frac{1}{2} q_1^5 - \frac{1}{4} q_1^2 q_3^2 + \frac{1}{2} q_2 q_3^2 + 2 q_1 q_2^2, \\ H_3 &= \frac{1}{2} p_3^2 q_1^2 + p_3^2 q_2 - p_1 p_3 q_3 - p_2 p_3 q_1 q_3 + \frac{1}{2} p_2^2 q_3^2 + \frac{1}{2} q_1^3 q_3^2 - q_1 q_2 q_3^2 - \frac{1}{8} q_3^4. \end{aligned}$$

However, as typically happens in the cases of Gelfand-Zakarevich systems [19], the reduced integrable Hamiltonian systems on  $S_0$  do not allow a bi-Hamiltonian formulation. Nevertheless, they can be described in the context of our new theory. In fact, proceeding analogously to the case of two degrees of freedom, we search for a Haantjes operator  $\mathbf{K}_1$  whose minimal polynomial be of the maximum degree allowed by our assumptions, in this case 3:

$$(164) \quad m_{\mathbf{K}_1}(\lambda) := \lambda^3 - c_1(\mathbf{x})\lambda^2 - c_2(\mathbf{x})\lambda - c_3(\mathbf{x}) ,$$

and that satisfy

$$(165) \quad \mathbf{K}_1^T \Omega = \Omega \mathbf{K}_1$$

$$(166) \quad \mathbf{K}_1^T dH_1 = dH_2$$

$$(167) \quad (\mathbf{K}_1^T)^3 dH_1 = (c_1(\mathbf{K}_1^T)^2 + c_2 \mathbf{K}_1^T + c_3 \mathbf{I}) dH_1$$

$$(168) \quad \mathcal{H}_{\mathbf{K}}(X, X') = 0 \quad \forall X, X' \in TM .$$

Under the simplest ansatz that its elements be *linear* in the Darboux coordinates  $(q_1, q_2, q_3, p_1, p_2, p_3)$ , we find the unique solution

$$(169) \quad \mathbf{K}_1 = \left[ \begin{array}{ccc|ccc} q_1 & 1 & 0 & 0 & 0 & 0 \\ -2q_2 & q_1 & -q_3 & 0 & 0 & 0 \\ -q_3 & 0 & 2q_1 & 0 & 0 & 0 \\ \hline 0 & -p_2 & -p_3 & q_1 & -2q_2 & -q_3 \\ p_2 & 0 & 0 & 1 & q_1 & 0 \\ p_3 & 0 & 0 & 0 & -q_3 & 2q_1 \end{array} \right] .$$

Since the minimal polynomial of  $\mathbf{K}_1$  is of degree 3, by virtue of Proposition 55  $\mathbf{K}_1$  is a generator of the Haantjes ring  $\mathcal{K}$ . Thus, we search for another Haantjes operator  $\mathbf{K}_2$  such that

$$(170) \quad \mathbf{K}_2 = f\mathbf{I} + g\mathbf{K}_1 + h\mathbf{K}_1^2 ,$$

$$(171) \quad \mathbf{K}_2^T dH_1 = dH_3 ,$$

where  $f, g, h$  are suitable smooth functions on  $M$ . The unique solution is  $\mathbf{K}_2 = (q_1^2 + 2q_2)\mathbf{I} - 2q_1\mathbf{K}_1 + \mathbf{K}_1^2$ , therefore

$$(172) \quad \mathbf{K}_2 = \left[ \begin{array}{ccc|ccc} 0 & 0 & -q_3 & 0 & 0 & 0 \\ q_3^2 & 0 & -q_1q_3 & 0 & 0 & 0 \\ -q_1q_3 & -q_3 & q_1^2 + 2q_2 & 0 & 0 & 0 \\ \hline 0 & 0 & p_2q_3 - p_3q_1 & 0 & q_3^2 & -q_1q_3 \\ 0 & 0 & -p_3 & 0 & 0 & -q_3 \\ -(p_2q_3 - p_3q_1) & p_3 & 0 & -q_3 & -q_1q_3 & q_1^2 + 2q_2 \end{array} \right] .$$

Since the  $\omega\mathcal{H}$  structure  $(M, \omega, \mathbf{I}, \mathbf{K}_1, \mathbf{K}_2)$  admits a maximal generator, i.e.,  $\mathbf{K}_1$ , the Hamiltonian functions (163) have a set of DH coordinates as separation variables. A set of such coordinates can be computed finding a Nijenhuis generator of the Haantjes module generated by  $\mathbf{K}_1$ . This is what one of the present authors did in [48] by a complete different method, that is to say by means of the Marsden-Ratiu reduction procedure above mentioned. In that paper, separation coordinates were called Darboux-Nijenhuis coordinates. In [28], the same coordinates have been considered as orthogonal separation variables for the Hamiltonian function (163), in the cotangent bundle of a three-dimensional Minkowski space.

**9.2. Separation of variables for a Drach-Holt type system.** The approach proposed in this paper offers an effective procedure to construct algorithmically separation variables. As a paradigmatic example, we shall study the case of a system showing an irrational dependence on its coordinates, namely a three-parametric deformation of the Holt potential, that has been introduced in [15]:

$$(173) \quad H_1 = \frac{1}{2} (p_x^2 + p_y^2) + a_1 \frac{4x^2 + 3y^2}{y^{2/3}} + a_2 \frac{x}{y^{2/3}} + \frac{a_3}{y^{2/3}} \quad a_1, a_2, a_3 \in \mathbb{R} .$$

This system is integrable in the manifold  $M = T^*(E^2 \setminus \{y = 0\})$ , with a third-order integral

$$(174) \quad \begin{aligned} H_2 = & 2p_x^3 + 3p_x p_y^2 + 12a_1 \left( 6xy^{1/3} p_y + \frac{2x^2 - 3y^2}{y^{2/3}} \right) + \\ & + a_2 \left( 9y^{1/3} p_y + \frac{6x}{y^{2/3}} p_x \right) + 6 \frac{a_3}{y^{2/3}} p_x . \end{aligned}$$

When  $a_1 \rightarrow 0$ , the Hamiltonian  $H_1$  converts into a one-parametric deformation of the Post-Winternitz superintegrable potential (157). A crucial aspect is that  $H_1$  has been considered in the literature to be an example of *nonseparable system*, since, not being of classical Stäckel type, it is not separable by means of an extended-point transformation. A natural question is whether there is a full canonical transformation (in general difficult to find) redeeming its separability. Our theory of integrability *à la Haantjes* enables us to solve this problem, since it provides a set of DH coordinates by means of the procedure outlined in Subsection 8.1.

**9.2.1. The Haantjes structure.** The prescription in step 1 provides with a Haantjes operator linear in the momenta, which reads

$$(175) \quad \mathbf{K}_{DH} = 3 \left[ \begin{array}{cc|cc} 2p_x & p_y & 0 & 3y \\ 0 & 2p_x & -3y & 0 \\ \hline 0 & -24a_1 y^{1/3} & 2p_x & 0 \\ 24a_1 y^{1/3} & 0 & p_y & 2p_x \end{array} \right] .$$

It endows the manifold  $M$  with the  $\omega H$  structure  $(\omega, \mathbf{I}, \mathbf{K}_{DH})$ . The Lenard-Haantjes chain is defined by

$$dH_2 = \mathbf{K}_{DH}^T dH_1 .$$

According to the general theory developed above, the potential functions of the exact one-forms belonging to the eigen-distributions of  $\mathbf{K}_{DH}^T$  provide the separation coordinates. In particular, as stated in Remark 70, it can be checked that this system is quasi-bi-Hamiltonian. Therefore the eigenvalues of the Haantjes operator (175) are just characteristic functions of the web of  $\mathbf{K}_{DH}$ , due to eq. (156).

**9.2.2. Separation Coordinates.** These functions, for  $a_1 > 0$ , read

$$(176) \quad \lambda_1 = l_2 = 6(p_x + 3\sqrt{2a_1}y^{2/3}), \quad \lambda_2 = l_1 = 6(p_x - 3\sqrt{2a_1}y^{2/3}) .$$

In order to get a system of DH coordinates, they can be completed with a pair of conjugate momenta that have the non trivial expressions

$$\begin{aligned}
(177) \quad \mu_1 &= \frac{1}{576} \sqrt{\frac{2}{a_1}} \left( p_x^4 + 12\sqrt{2a_1} p_x^3 y^{2/3} + 108a_1 p_x^2 y^{4/3} + 216\sqrt{2a_1^3} p_x y^2 \right. \\
&\quad \left. + 12p_y y^{1/3} - 24\sqrt{2a_1} x + 32a_1^2 y^{8/3} \right), \\
\mu_2 &= \frac{1}{576} \sqrt{\frac{2}{a_1}} \left( p_x^4 - 12\sqrt{2a_1} p_x^3 y^{2/3} + 108a_1 p_x^2 y^{4/3} - 216\sqrt{2a_1^3} p_x y^2 \right. \\
&\quad \left. - 12p_y y^{1/3} - 24\sqrt{2a_1} x + 32a_1^2 y^{8/3} \right),
\end{aligned}$$

and have been computed as potential functions of two exact 1-forms belonging to the two characteristic eigen-distributions  $(E_1^\circ, E_2^\circ)$  of  $\mathbf{L}_{DH}^T$ , that locally decompose  $T^*M = E_1^\circ \oplus E_2^\circ$ . The set of coordinates (176), (177) are separation variables for the Hamiltonian functions  $H_1$  and  $H_2$ , due to Theorem 59.

**9.2.3. Separation Equations of Jacobi–Sklyanin.** In order to solve the Hamilton–Jacobi equation of the Drach–Holt model, one has to write down the Jacobi–Sklyanin equations that read

$$\begin{aligned}
b_1 \mu_1^2 - (b_2 \lambda_1^4 + b_3) \mu_1 + b_4 \lambda_1^8 + b_5 \lambda_1^4 + b_6 \lambda_1^3 - \lambda_1 H_1 + H_2 + b_7 &= 0, \\
b_1 \mu_2^2 - (b_2 \lambda_2^4 + b_3) \mu_2 + b_4 \lambda_2^8 + b_5 \lambda_2^4 - b_6 \lambda_2^3 + \lambda_2 H_1 - H_2 + b_7 &= 0,
\end{aligned}$$

where  $b_i$ ,  $i = 1, \dots, 7$  are the constants given by

$$\begin{aligned}
b_1 &= 10368 \sqrt{2a_1^3}, \quad b_2 = \frac{a_1}{18}, \quad b_3 = 216 a_2 \sqrt{2a_1}, \quad b_4 = \frac{\sqrt{2a_1}}{26873856}, \\
b_5 &= \frac{a_2}{1728}, \quad b_6 = \frac{1}{216}, \quad b_7 = 18 a_3 \sqrt{2a_1}.
\end{aligned}$$

We arrive therefore at the separated solutions of the Hamilton–Jacobi equation

$$\begin{aligned}
W_1(\lambda_1; h_1, h_2) &= \frac{1}{2b_1} \int^{\lambda_1} \left( - (b_2 \lambda_1^4 + b_3) \pm \sqrt{(b_2 \lambda_1^4 + b_3)^2 - 4b_1 P_8(\lambda_1)} \right) d\lambda_1, \\
W_2(\lambda_2; h_1, h_2) &= \frac{1}{2b_1} \int^{\lambda_2} \left( - (b_2 \lambda_2^4 + b_3) \pm \sqrt{(b_2 \lambda_2^4 + b_3)^2 - 4b_1 Q_8(\lambda_2)} \right) d\lambda_2,
\end{aligned}$$

where  $h_1, h_2$  are the values of  $H_1, H_2$  on the lagrangian tori, and

$$\begin{aligned}
P_8(\lambda_1; h_1, h_2) &:= b_4 \lambda_1^8 + b_5 \lambda_1^4 + b_6 \lambda_1^3 - \lambda_1 h_1 + h_2 + b_7, \\
Q_8(\lambda_2; h_1, h_2) &:= b_4 \lambda_2^8 + b_5 \lambda_2^4 - b_6 \lambda_2^3 + \lambda_2 h_1 - h_2 + b_7.
\end{aligned}$$

## 10. FUTURE PERSPECTIVES

The extension of the present theory to the case of quantum integrable systems is a nontrivial task. This research line would pave the way to an algebraic interpretation of the notion of Haantjes integrability developed here, in terms of infinite-dimensional commuting operators on a Hilbert space.

Also, it would be interesting to compare the geometric structures underlying the vision offered here with the intrinsic, purely algebraic structures developed in [31], in the context of *nilpotent integrability*.

A natural extension of the present theory to the case of *superintegrable systems* [59], especially maximally superintegrable ones, is in order. Along these lines, we also wish to construct a generalization of our approach to the study of the geometry

of certain classical systems, as the Post-Winternitz model of Section 8.2, that do not possess any simple system of separation coordinates. We believe that our theory can offer a proper language in which the study of the relation between superintegrability and separability indeed can be carried out.

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